Lecture 22

Financial derivatives and PDE's
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Forwards

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a type of financial derivative stipulated by two parties in which one of the parties promises to sell and deliver to the other party the asset $\mathcal{U}$ at time T in exchange for the cash K.

The party who agrees to buy, resp. sell, the asset has the long, resp. short, position on the forward contract.

Note that, as opposed to option contracts, both parties in a forward contract are *obliged* to fulfill their part of the agreement. Forward contracts are traded over the counter and most commonly on commodities or currencies. Let us give two examples.

Example of forward contract on a commodity. Consider a farmer who grows wheat and a miller who needs wheat to produce flour. Clearly, the farmer interest is to sell the wheat for the highest possible price, while the miller interest is to pay the least possible for the wheat. The price of the wheat depends on many economical and non-economical factors (such as whether conditions, which affect the quality and quantity of harvests) and it is therefore quite volatile. The farmer and the miller then stipulate a forward contract on the wheat in the winter (before the plantation, which occurs in the spring) with expiration date in the end of the summer (when the wheat is harvested), in order to lock its future trading price beforehand.

Example of forward contract on a currency. Suppose that a car company in Sweden promises to deliver a stock of 100 cars to another company in the United States in exactly one month. Suppose that the price of each car is fixed in Swedish crowns, say 100.000 crowns. Clearly the American company will benefit by an increase of the exchange rate crown/dollars and will be damaged in the opposite case. To avoid possible high losses, the American company by a forward contract on 100×100.000=ten millions Swedish crowns expiring in one month which gives the company the right *and* the obligation to buy ten millions crowns for a price in dollars agreed upon today.

The delivery price K in a forward contract is the result of a pondered estimation for the price of the asset at the time T in the future.

In this respect, K is also called the **forward price** of $\mathcal{U}$.

More precisely, the T-forward price of the asset $\mathcal{U}$ at time t is the strike price of a forward contract on $\mathcal{U}$ with maturity T stipulated at time t; the current, actual price $\Pi(t)$ of the asset is also called the **spot** price.

Let us apply the risk-neutral pricing theory to derive a mathematical model for the forward price of financial assets.

Let $f(t, \Pi(t), K, T)$ be the value at time t of the forward contract on the asset $\mathcal{U}$ with maturity T and delivery price K.

Here $\{\Pi(t)\}_{t\in[0,T]}$ is the price process of the underlying asset. The pay-off for the long position on the forward is given by

$$Y_{\text{long}} = (\Pi(T) - K),$$

while the pay-off for the short position is $Y_{\text{short}} = (K - \Pi(T))$.

As both positions entail the same rights/obligations, and thus no privileged position exists, none of the two parties has to pay a premium to stipulate the forward contract.

Hence $f(t, \Pi(t), K, T) = 0$.

$d\Pi(t) = r(t)\Pi(t)\,dt + \sigma(t)\Pi(t)\,d\widetilde{W}(t)$

Assuming that the price $\{\Pi(t)\}_{t\geq 0}$ of the underlying asset follows a generalized geometric Brownian motion with strictly positive volatility, the risk-neutral value of the forward contract for the two parties is

$\frac{D(T)}{D(t)}$

$$\begin{aligned} f(t, \Pi(t), K, T) &= \pm\widetilde{\mathbb{E}}[(\Pi(T) - K)D(T)/D(t)|\mathcal{F}_W(t)] \\ &= \pm\left(\frac{1}{D(t)}\underbrace{\widetilde{\mathbb{E}}(\Pi(T)D(T)|\mathcal{F}_W(t))}_{=D(t)\Pi(t)} - K\widetilde{\mathbb{E}}[\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)]\right). \end{aligned}$$

$B(t,T)$

As the discounted price $\{\Pi^*(t)\}_{t\geq 0}$ of the underlying asset is a $\widetilde{\mathbb{P}}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, then $\widetilde{\mathbb{E}}(\Pi(T)D(T)|\mathcal{F}_W(t)) = D(t)\Pi(t)$. Hence

$$f(t,\Pi(t),K,T) = \pm(\Pi(t) - \boxed{K}B(t,T)), \quad = 0$$

where
$$B(t,T) = \widetilde{\mathbb{E}}[\exp(-\int_t^T r(s)\,ds)|\mathcal{F}_W(t)], \tag{1}$$

is the value of the ZCB computed using the risk-neutral probability. This leads to the following definition.

Definition 1. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of the asset satisfies*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t),$$

where $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0}, \{r(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ and $\sigma(t) > 0$ almost surely for all times. The **risk-neutral T-forward price** *at time t of the asset is the $\{\mathcal{F}_W(t)\}_{t\geq 0}$-adapted stochastic process $\{\mathrm{For}_T(t)\}_{t\in[0,T]}$ given by*

$$\mathrm{For}_T(t) = \frac{\Pi(t)}{B(t,T)}, \quad t \in [0,T],$$

where $B(t,T)$ is given by (1).

Note that, as long as the spot rate is positive, the forward price increases with respect to the time left to delivery, i.e., the longer we delay the delivery of the asset, the more we have to pay for it. This is intuitive, as the seller of the asset is loosing money by not selling the asset on the spot (due to its devaluation compared to the bond value).

Remark For a constant interest rate $r(t) = r$ the forward price becomes

$$\mathrm{For}_T(t) = e^{r\tau}\Pi(t), \quad \tau = T - t,$$

in which case we find that the spot price of an asset is the discounted value of the forward price. When the asset is a commodity (e.g., corn), the forward price is also inflated by the cost of storage. Letting $c > 0$ be the cost to storage one share of the asset for one year, then the forward price of the asset, for delivery in τ years in the future, is $e^{c\tau}e^{r\tau}\Pi(t)$.

Forward probability measure } CHANGE OF "NUMERAIRE"

Let $t \in [0,T]$ and define

$$B^*(t,T) = \tilde{\mathbb{E}}[D(T)|\mathcal{F}_W(t)]$$

$$Z^{(T)}(t) = \frac{D(t)B(t,T)}{B(0,T)} = \frac{B^*(t,T)}{B(0,T)}. \tag{2}$$

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$$Z^{(T)}(0) = 1$$

$\{Z^{(T)}(t)\}_{t\in[0,T]}$ is $\tilde{\mathbb{P}}$-MARTINGALE $\Rightarrow \tilde{\mathbb{E}}[Z^{(T)}(t)] = 1$

$$\tilde{\mathbb{P}}(A) = \mathbb{E}[Z(T)\mathbb{I}_A] \qquad Z(t) = e^{-\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds} \qquad \theta(t) = \frac{\alpha(t) - r(t)}{\sigma(t)}$$

$$\mathbb{P}^{(T)}(A) = \tilde{\mathbb{E}}[Z^{(T)}(t)\mathbb{I}_A] \qquad Z^{(T)}(t) = \frac{D(t)B(t,T)}{B(0,T)} = e^{-\int_0^t \tilde{\theta}(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^t \tilde{\theta}(s)^2 ds}$$

WHERE $\tilde{\theta}(t)$ IS GIVEN BELOW IN THE NEXT PAGE

As the discounted value of the ZCB is a martingale in the risk-neutral probability measure, then the process $\{Z^{(T)}(t)\}_{t\in[0,T]}$ is also a $\widetilde{\mathbb{P}}$-martingale. Moreover $\widetilde{\mathbb{E}}[Z^{(T)}(t)] = 1$, hence, by Theorem (??), for all $t \in [0,T]$, the function $\mathbb{P}^{(T)}(A) = \widetilde{\mathbb{E}}[Z^{(T)}(t)\mathbb{I}_A]$ defines a probability measure equivalent to $\widetilde{\mathbb{P}}$, which is called the ***T*-forward probability measure**.

Theorem 1. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of the asset satisfies*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t), \quad = r(t)\Pi(t)dt + \sigma(t)\Pi(t)d\tilde{W}(t)$$

where $\{\alpha(t)\}_{t\geq 0}, \{\sigma(t)\}_{t\geq 0}, \{r(t)\}_{t\geq 0} \in \mathcal{C}^0[\mathcal{F}_W(t)]$ and $\sigma(t) > 0$ almost surely for all times. The risk-neutral T-forward price process $\{\mathrm{For}_T(t)\}_{t\in[0,T]}$ of the asset is a martingale in the T-forward probability measure relative to the filtration $\{\mathcal{F}_W(t)\}$ and

$$\boxed{\mathrm{For}_T(t) = \mathbb{E}^{(T)}[\Pi(T)|\mathcal{F}_W(t)].} \tag{3}$$

Moreover risk-neutral pricing formula (??) for the European derivative with pay-off Y at maturity T can be written in terms of the forward probability measure as follows:

$$\Pi_Y(t) = \frac{1}{D(t)}\tilde{\mathbb{E}}[D(T)Y|\mathcal{F}_W(t)]$$

$$\Pi_Y(t) = B(t,T)\mathbb{E}^{(T)}[Y|\mathcal{F}_W(t)]. \tag{4}$$

maturity T can be written in terms of the forward probability measure as follows:

$$\Pi_Y(t) = B(t,T)\mathbb{E}^{(T)}[Y|\mathcal{F}_W(t)] \tag{4}$$

$$\Pi_Y(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[D(T)Y|\mathcal{F}_W(t)]$$

Proof. By ~~(??)~~ (3.24) we have

$$\begin{aligned}\mathbb{E}^{(T)}[\Pi(T)|\mathcal{F}_W(t)] &= \frac{1}{Z^{(T)}(t)}\widetilde{\mathbb{E}}[Z^{(T)}(T)\Pi(T)|\mathcal{F}_W(t)]\\ &= \frac{B(0,T)}{D(t)B(t,T)}\widetilde{\mathbb{E}}[D(T)\Pi(T)/B(0,T)|\mathcal{F}_W(t)].\end{aligned}$$

$D(t)\,\Pi(t)$

$$Z^{(T)}(t) = \frac{D(t)B(t,T)}{B(0,T)}$$

$$Z^{(T)}(T) = \frac{D(T)}{B(0,T)}$$

As the discounted value of the asset is a martingale in the risk-neutral probability, we have

$$\mathbb{E}^{(T)}[\Pi(T)|\mathcal{F}_W(t)] = \frac{\Pi(t)}{B(t,T)} = \mathrm{For}_T(t).$$

Moreover, by Exercise ~~??~~ 3.30, equation (3) implies that the forward price is a $\mathbb{P}^{(T)}$-martingale relative to the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Similarly, By (3.24)

$$\mathbb{E}^{(T)}[Y|\mathcal{F}_W(t)] = \frac{1}{Z^{(T)}(t)}\widetilde{\mathbb{E}}[Z^{(T)}(T)Y|\mathcal{F}_W(t)] = \frac{1}{B(t,T)}\widetilde{\mathbb{E}}[D^{-1}(t)D(T)Y|\mathcal{F}_W(t)]$$

$\Pi_Y(t)$

and so (4) follows. □

An advantage of writing the risk-neutral pricing formula in the form (4) is that the interest rate process does not appear within the conditional expectation.

Thus, while in the risk-neutral probability measure one needs the joint distribution of the discount factor $D(T)$ and the pay-off Y in order to compute the price of the derivative, in the T-forward probability measure only the distribution of the pay-off is required.

THEOREM 6.2(ii): THERE EXISTS A STOCHASTIC PROCESS $\{\Gamma(t)\}_{t\geq 0}$ ADAPTED TO $\{\mathcal{F}_W(t)\}_{t\geq 0}$, SUCH THAT $d\Pi^*_Y(t) = \Gamma(t)\, d\widetilde{W}(t)$

IN PARTICULAR, $dB^*(t,T) = \Gamma(t)\, d\widetilde{W}(t) \Rightarrow dZ^{(T)}(t) = \Delta(t)\, d\widetilde{W}(t)$

This suggests a method to compute the risk-neutral price of a derivative when the market parameters are stochastic.

$$Z^{(T)}(t) = \frac{B^*(t,T)}{B(0,T)}$$

Before we see an example of application of this argument, we remark that the process $\{Z_T(t)\}_{t\in[0,T]}$ satisfies $dZ^{(T)}(t) = \Delta(t)Z^{(T)}d\widetilde{W}(t)$, for some process $\{\Delta(t)\}_{t\in[0,T]}$ adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$. Hence

THIS FOLLOWS BY THEOREM 6.2(ii)

WHERE $\Delta(t) = \frac{\Gamma(t)}{B(0,T)}$

$$\boxed{Z^{(T)}(t) = \exp\left(-\int_0^t \widetilde{\theta}(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \widetilde{\theta}^2(s)\,ds\right), \quad \widetilde{\theta}(t) = -\Delta(t).}$$

It follows by Girsanov's theorem that the process $\{W^{(T)}(t)\}_{t\in[0,T]}$ given by

$$d(\log Z^{(T)}(t)) = \frac{1}{Z^{(T)}(t)}dZ^{(T)}(t) - \frac{dZ^{(T)}(t)\,dZ^{(T)}(t)}{2Z^{(T)}(t)^2}$$

$$W^{(T)}(t) = \widetilde{W}(t) + \int_0^t \widetilde{\theta}(s)\,ds \tag{5}$$

is a Brownian motion in the T-forward measure.

Theorem 2. *Assume that the price $\{\Pi(t)\}_{t\geq 0}$ of the asset $\mathcal{U}$ satisfies*

$$d\Pi(t) = \alpha(t)\Pi(t)dt + \sigma(t)\Pi(t)dW(t), \quad \Pi(0) = \Pi_0 > 0, \quad \sigma(t) > 0, \; t \in [0,T],$$

and that there exists a constant $\widehat{\sigma} > 0$ such that

$$d\mathrm{For}_T(t) = \widehat{\sigma}\mathrm{For}_T(t)dW^{(T)}(t), \quad \mathrm{For}_T(0) = F_0 := \Pi_0/B(0,T), \tag{6}$$

where $\{W^{(T)}(t)\}_{t\in[0,T]}$ is the $\mathbb{P}^{(T)}$-Brownian motion defined above. Then the value at time $t=0$ of the European call on the asset $\mathcal{U}$ with strike K and maturity T can be written as

$$\Pi_{\mathrm{call}}(0) = \Pi_0\Phi(d_+) - KB(0,T)\Phi(d_-), \tag{7}$$

IN BLACK-SCHOLES $B(0,T) \to e^{-rT}$

where

$$d_\pm = \frac{\log\left(\frac{\Pi_0}{KB(0,T)}\right) \pm \frac{1}{2}\widehat{\sigma}^2 T}{\widehat{\sigma}\sqrt{T}}. \tag{8}$$

Proof. According to (4) we have

$$\Pi_{\mathrm{call}}(0) = B(0,T)\mathbb{E}^{(T)}[(\overbrace{\Pi(T)}^{Y} - K)_+]. = B(0,T)\,\mathbb{E}^{(T)}[(F_0 e^{-\frac{1}{2}\hat{\sigma}^2 T - \hat{\sigma}\sqrt{T}G} - K)_+]$$

In the right hand side we replace

$$\Pi(T) = \mathrm{For}_T(T) = F_0 e^{-\frac{1}{2}\widehat{\sigma}^2 T - \widehat{\sigma}W^{(T)}(T)} = F_0 e^{-\frac{1}{2}\widehat{\sigma}^2 T - \widehat{\sigma}\sqrt{T}G},$$

and compute the expectation using that $G \in \mathcal{N}(0,1)$ in the forward measure, that is

$$\Pi_{\mathrm{call}}(0) = B(0,T)\int_{\mathbb{R}}(F_0 e^{-\frac{1}{2}\widehat{\sigma}^2 T - \widehat{\sigma}\sqrt{T}y} - K)_+ e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}.$$

The result now follows by computing the integral as in the proof of Theorem ~~2~~ 6.6. $\square$