Lecture 26-27

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Financial derivatives and PDE's
Lectures 26-27

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March 3^{rd}-4^{th}, 2021

$Y(t) = (S(t) - K)_+ = g(S(t))$; $g(x) = (x-K)_+$ FOR AN AMERICAN CALL OPTION

1 Introduction to American derivatives

Before giving the precise definition of fair price for American derivatives, we shall present some general properties of these contracts.

American derivatives can be exercised at any time prior or including maturity T. Let $Y(t)$ be the pay-off resulting from exercising the derivative at time $t \in (0, T]$.

We call $Y(t)$ the **intrinsic value** of the derivative. We consider only standard American derivatives, for which we have $Y(t) = g(S(t))$, for some measurable function $g : \mathbb{R} \to \mathbb{R}$.

For instance, $g(x) = (x - K)_+$ for American calls and $g(x) = (K - x)_+$ for American puts.

$\widehat{\Pi}_Y(t) = ?$

We denote by $\widehat{\Pi}_Y(t)$ the rsik-neutral price of the American derivative with intrinsic value $Y(t)$ and by $\Pi_Y(t)$ the risk-neutral price of the European derivative with pay-off $Y = Y(T)$ at maturity time T (given by (22)).

$\Pi_Y(t) = \widetilde{\mathbb{E}}\left[\frac{D(T)}{D(t)} Y \mid \mathcal{F}_W(t)\right]$

Even if we do not know yet how $\widehat{\Pi}_Y(t)$ is defined, two obvious properties of American derivatives are the following:

(i) $\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$, for all $t \in [0, T]$. In fact an American derivative gives to its owner all the rights of the corresponding European derivative plus one: the option of early exercise. Thus it is clear that the American derivative cannot be cheaper than the European one.

(ii) $\widehat{\Pi}_Y(t) \geq Y(t)$, for all $t \in [0, T]$. If not, an arbitrage opportunity would arise by purchasing the American derivative and exercising it immediately.

$\widehat{\Pi}_Y(T) = \Pi_Y(T) = Y(T)$

IF $\widehat{\Pi}_Y(t) > Y(t)$

IF BUY AT $t=0$ FOR $\widehat{\Pi}_Y(0)$ AND CLOSE THE POSITION AT TIME t

IF $\widehat{\Pi}_Y(t) > Y(t)$

IF BUY AT $t=0$ FOR $\widehat{\Pi}_Y(0)$ AND CLOSE THE POSITION AT TIME t THEN THE RETURN IT $\widehat{\Pi}(t) - \widehat{\Pi}(0)$ IF SELL THE DERIVATIVE, $Y(t) - \widehat{\Pi}(0)$ IF EXERCISE THE DERIVATIVE

Any reasonable definition of fair price for American derivatives must satisfy (i)-(ii).

Definition 1. *A time $t \in (0, T]$ is said to be an* **optimal exercise time** *for the American derivative with intrinsic value $Y(t)$ if $\widehat{\Pi}_Y(t) = Y(t)$.*

Hence by exercising the derivative at an optimal exercise time t, the buyer takes full advantage of the derivative: the resulting pay-off equals the value of the derivative.

On the other hand, if $\widehat{\Pi}_Y(t) > Y(t)$ and the buyer wants to close the (long) position on the American derivative, then the optimal strategy is to sell the derivative, thereby cashing the amount $\widehat{\Pi}_Y(t)$.

$Y(t) = (S(t) - K)_+$

Theorem 1. *Assume (i) holds and let $\widehat{C}(t)$ be the price of an American call at time $t \in [0, T]$. Assume further that the underlying stock price follows a generalized geometric Brownian motion and that the interest rate $r(t)$ of the money market is strictly positive for all times. Then $\widehat{C}(t) > Y(t)$, for all $t \in [0, T)$. In particular it is never optimal to exercise the call prior to maturity.*

IN PARTICULAR, $\widehat{C}(t) = C(t)$

Proof. For $S(t) \le K$ the claim becomes $\widehat{C}(t) > 0$ for $t \in [0, T)$, which is obvious (since $\widehat{C}(t) \ge C(t) > 0$). For $S(t) > K$ we write

$$\begin{aligned}\widehat{C}(t) \ge C(t) &= \widetilde{\mathbb{E}}[(S(T) - K)_+ D(T)/D(t)|\mathcal{F}_W(t)] \ge \widetilde{\mathbb{E}}[(S(T) - K)D(T)/D(t)|\mathcal{F}_W(t)] \\ &= \widetilde{\mathbb{E}}[S(T)D(T)/D(t)|\mathcal{F}_W(t)] - K\underbrace{\widetilde{\mathbb{E}}[D(T)/D(t)|\mathcal{F}_W(t)]}_{B(t,T)\,<\,1} > D(t)^{-1}\widetilde{\mathbb{E}}[S^*(T)|\mathcal{F}_W(t)] - K \\ &= S(t) - K = (S(t) - K)_+ \le Y(t)\end{aligned}$$

$S^*(t) = D(t)S(t)$

where we used $D(T)/D(t) < 1$ (by the positivity of the interest rate $r(t)$) and the martingale property of the discounted price $\{S^*(t)\}_{t \in [0,T]}$ of the stock. $\square$

It follows that under the assumptions of the previous theorem the earlier exercise option of the American call is worthless, hence *American and European call options with the same strike and maturity have the same value.*

Remark. A notable exception to the assumed conditions in Theorem 1 is when the underlying stock pays a dividend. In this case it can be shown that it is optimal to exercise the American call immediately before the dividend is payed, provided the price of the stock is sufficiently high.

Definition 2. *Let $T \in (0, \infty)$. A random variable $\tau : \Omega \to [0, T]$ is called a* **stopping time** *for the filtration $\{\mathcal{F}_W(t)\}_{t \ge 0}$ if $\{\tau \le t\} \in \mathcal{F}_W(t)$, for all $t \in [0, T]$. We denote by Q_T the set of all stopping times for the filtration $\{\mathcal{F}_W(t)\}_{t \ge 0}$.*

Think of τ as the time at which some random event takes place.

$\{\tau \le t\} = \{\omega \in \Omega : \tau(\omega) \le t\}$

Then τ is a stopping time if the occurrence of the event before or at time t can be inferred by the information available up to time t (no future information is required).

For the applications that we have in mind, τ will be the optimal exercise time of an American derivative, which marks the event that the price of the derivative equals its intrinsic value.

From now on we assume that the market has constant parameters and $r > 0$. Hence the price of the stock is given by the geometric Brownian motion

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma\widetilde{W}(t)}.$$

We recall that in this case the price $\Pi_Y(0,T)$ at time $t = 0$ of the European derivative with pay-off $Y = g(S(T))$ at maturity time $T > 0$ is given by

$$\Pi_Y(0,T) = \widetilde{\mathbb{E}}[e^{-rT}g(S(T))].$$

$Y(t) = g(S(t))$

Now, if the writer of the American derivative were sure that the buyer would exercise at the time $u \in (0,T]$, then the fair price of the American derivative at time $t = 0$ would be equal to $\Pi_Y(0,u)$.

$\Pi_Y(0,u) = \widetilde{\mathbb{E}}[e^{-ru} g(S(u))]$

As the writer cannot anticipate when the buyer will exercise, we would be tempted to define the price of the American derivative at time zero as $\max\{\cancel{\Pi_Y(0,u), 0 \leq u} \leq T\}$.

However this definition would actually be unfair, as it does not take into account the fact that the exercise time is a stopping time, i.e., it is random and it cannot be inferred using future information.

This leads us to the following definition.

Definition 3. *In a market with constant parameters, the risk-neutral (or fair) price at time $t = 0$ of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ and maturity $T > 0$ is given by*

$$\widehat{\Pi}_Y(0) = \max_{\tau \in Q_T} \widetilde{\mathbb{E}}[e^{-r\tau}g(S(\tau))], \tag{1}$$

where $S(\tau) = S(0)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\widetilde{W}(\tau)}$.

$S(\tau(\omega)) = S(0)e^{(r-\frac{\sigma^2}{2})\tau(\omega)+\sigma\widetilde{W}(\tau(\omega))}$

It is not possible in general to find an closed formula for the risk-neutral price of American derivatives. A notable exception is the price of perpetual American put options, which we discuss next.

Perpetual American put options

An American put option is called perpetual if it never expires, i.e., $T = \infty$. This is of course an idealization, but perpetual American puts are very useful to visualize the structure of general American put options.

Definition 1 becomes the following.

WITH VALUE IN THE INTERVAL $[0, \infty]$

Definition 4. *Let Q be the set of all stopping times for the filtration $\{\mathcal{F}_W(t)\}_{t\geq 0}$, i.e., $\tau \in Q$ iff $\tau : \Omega \to [0, \infty]$ is a random variable and $\{\tau \leq t\} \in \mathcal{F}_W(t)$, for all $t \geq 0$. The risk-neutral price at time $t = 0$ of the perpetual American put with strike K is*

$g(x) = (K - x)_+$

$$\widehat{\Pi}(0) = \max_{\tau \in Q} \widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+] \qquad (2)$$

where $S(\tau) = S(0)e^{(r-\frac{\sigma^2}{2})\tau + \sigma\widetilde{W}(\tau)}$.

$U : Q \to (0, \infty)$

Theorem 2. *There holds*

$$\widehat{\Pi}(0) = v_L(S(0)), \qquad (3)$$

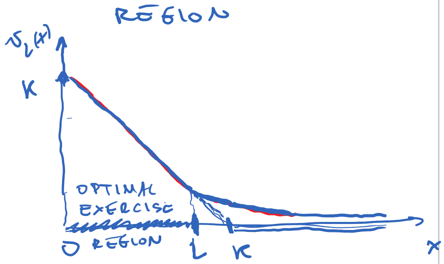
where

$$v_L(x) = \begin{cases} K - x & 0 \leq x \leq L \\ (K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x > L \end{cases}$$

OPTIMAL EXERCISE REGION

and

$$L = \frac{2r}{2r + \sigma^2}K.$$



Before we prove the theorem, some remarks are in order:

(i) $L < K$;

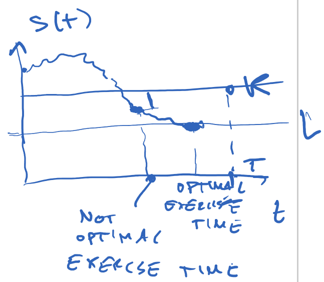
(ii) For $S(0) \leq L$ we have $\widehat{\Pi}(0) = v_L(S(0)) = K - S(0) = (K - S(0))_+$. Hence when $S(0) \leq L$ it is optimal to exercise the derivative.

(iii) We have $\widehat{\Pi}(0) > (K - S(0))_+$ for $S(0) > L$. In fact

$$v_L'(x) = -\frac{2r}{\sigma^2}\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}-1}\frac{K - L}{L},$$

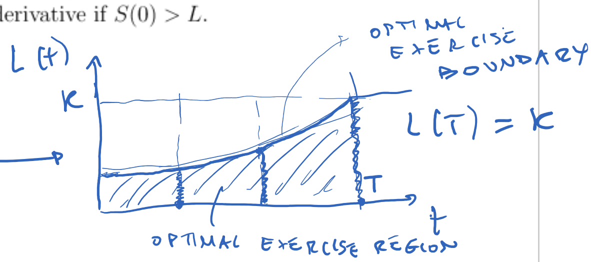
hence $v_L'(L) = -1$. Moreover

$$v_L''(x) = \frac{2r}{\sigma^2}\left(\frac{2r}{\sigma^2} + 1\right)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}-2}\frac{K - L}{L^2},$$

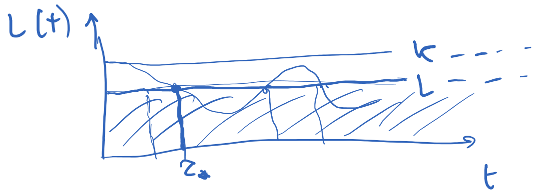


which is always positive. Thus the graph of $v_L(x)$ always lies above $K - x$ for $x > L$. It follows that it is not optimal to exercise the derivative if $S(0) > L$.

AMERICAN PUT WITH MATURITY T



PERPETUAL AMERICAN PUT



(iv) In the perpetual case, any time is equivalent to $t = 0$, as the time left to maturity is always infinite. Hence

$$\boxed{\widehat{\Pi}(t) = v_L(S(t)).}$$

In conclusion the theorem is saying us that the buyer of the derivative should exercise as soon as the stock price falls below the threshold L. In fact we can reformulate the theorem in the following terms:

Theorem 3. *The maximum of* $\widetilde{\mathbb{E}}[e^{-r\tau}(K - S(\tau))_+]$ *over all possible* $\tau \in Q$ *is achieved at* $\tau = \tau_*$, *where*

$$\tau_* = \min\{t \geq 0 : S(t) = L\}.$$

Moreover $\widetilde{\mathbb{E}}[e^{-r\tau_*}(K - S(\tau_*))_+] = v_L(S(0))$.

For the proof of Theorem 2 we need the optional sampling theorem:

Theorem 4. *Let* $\{X(t)\}_{t\geq 0}$ *be an adapted process and* τ *a stopping time. Let* $t \wedge \tau = \min(t, \tau)$. *If* $\{X(t)\}_{t\geq 0}$ *is a martingale/supermatingale/submartingale, then* $\{X(t\wedge\tau)\}_{t\geq 0}$ *is also a martingale/supermartingale/submartingale.*

We can now prove Theorem 2. We divide the proof in two steps, which correspond respectively to Theorem 8.3.5 and Corollary 8.3.6 in [?].

Step 1: *The stochastic process* $\{e^{-r(t\wedge\tau)}v_L(S(t\wedge\tau))\}_{t\geq 0}$ *is a super-martingale for all* $\tau \in Q$. *Moreover for* $S(0) > L$ *the stochastic process* $\{e^{-r(t\wedge\tau_*)}v_L(S(t\wedge\tau_*))\}_{t\geq 0}$ *is a martingale.* By Itô's formula,

$$d(e^{-rt}v_L(S(t))) = e^{-rt}[-rv_L(S(t)) + rS(t)v_L'(S(t)) + \frac{1}{2}\sigma^2 S(t)^2 v_L''(S(t))]dt + e^{-rt}\sigma S(t)v_L'(S(t))d\widetilde{W}(t).$$

The drift term is zero for $S(t) > L$ and it is equal to $-rK\,dt$ for $S(t) \leq L$. Hence

$$e^{-rt}v_L(S(t)) = v_L(S(0)) - rK\int_0^t e^{-ru}\mathbb{I}_{S(u)\leq L}(u)\,du + \int_0^t e^{-ru}\sigma S(u)v_L'(S(u))d\widetilde{W}(u).$$

Since the drift term is non-positive, then $\{e^{-rt}v_L(t)\}_{t\geq 0}$ is a supermartingale and thus by the optional sampling theorem, the process $\{e^{-r(t\wedge\tau)}v_L(S(t\wedge\tau)\}_{t\geq 0}$ is a also a supermartingale, for all $\tau \in Q$. Now, if $S(0) > L$, then, by continuity of the paths of the geometric Brownian motion, $S(u,\omega) > L$ as long as $u < \tau_*(\omega)$. Hence by stopping the process at τ_* the stock price will never fall below L and therefore the drift term vanishes, that is

$$e^{-r(t\wedge\tau_*)}v_L(S(t\wedge\tau_*)) = v_L(S(0)) + \int_0^{t\wedge\tau_*} e^{-ru}\sigma S(u)v_L'(S(u))d\widetilde{W}(u).$$

The Itô integral is a martingale and thus the Itô integral stopped at time τ_* is also a martingale by the optional sampling theorem. The claim follows.

Step 2: *The identity* (3) *holds.* The supermartingale property of the process $\{e^{-r(t\wedge\tau)}v_L(S(t\wedge\tau))\}_{t\geq 0}$ implies that its expectation is non-increasing, hence

$$\widetilde{\mathbb{E}}[e^{-r(t\wedge\tau)}v_L(S(t\wedge\tau))] \leq v_{L*}(S(0)).$$

As $v_L(x)$ is bounded and continuous, the limit $t \to +\infty$ gives

$$\widetilde{\mathbb{E}}[e^{-r\tau}v_L(S(\tau))] \leq v_L(S(0)).$$

As $v_L(x) \geq (K-x)_+$ we also have

$$\widetilde{\mathbb{E}}[e^{-r\tau}(K-S(\tau))_+] \leq v_L(S(0)).$$

Taking the maximum over all $\tau \in Q$ we obtain

$$\widehat{\Pi}(0) = \max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K-S(\tau))_+] \leq v_L(S(0)).$$

Now we prove the reverse inequality $\widehat{\Pi}(0) \geq v_L(S(0))$. This is obvious for $S(0) \leq L$. In fact, letting $\widetilde{\tau} = \min\{t \geq 0 : S(t) \leq L\}$, we have $\widetilde{\tau} \equiv 0$ for $S(0) \leq L$ and so $\max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K-S(\tau))_+] \geq \widetilde{\mathbb{E}}[e^{-r\widetilde{\tau}}(K-S(\widetilde{\tau}))_+] = (K-S(0))_+ = v_L(S(0))$, for $S(0) \leq L$. For $S(0) > L$ we use the martingale property of the stochastic process $\{e^{-r(t\wedge\tau_*)}v_L(S(t\wedge\tau_*))\}_{t\geq 0}$, which implies

$$\widetilde{\mathbb{E}}[e^{-r(t\wedge\tau_*)}v_L(S(t\wedge\tau_*))] = v_L(S(0)).$$

Hence in the limit $t \to +\infty$ we obtain

$$v_L(S(0)) = \widetilde{\mathbb{E}}[e^{-r\tau_*}v_L(S(\tau_*))].$$

Moreover $e^{-r\tau_*}v_L(S(\tau_*)) = e^{-r\tau_*}v_L(L) = e^{-r\tau_*}(K-S(\tau_*))_+$, hence

$$v_L(S(0)) = \widetilde{\mathbb{E}}[e^{-r\tau_*}(K-S(\tau_*))_+].$$

It follows that

$$\widehat{\Pi}(0) = \max_{\tau\in Q}\widetilde{\mathbb{E}}[e^{-r\tau}(K-S(\tau))_+] \geq v_L(S(0)),$$

which completes the proof. □

Next we discuss the problem of hedging the perpetual American put with a portfolio invested in the underlying stock and the risk-free asset.

Definition 5. *A portfolio process $\{h_S(t), h_B(t)\}_{t\geq 0}$ is said to be replicating the perpetual American put if its value $\{V(t)\}_{t\geq 0}$ equals $\widehat{\Pi}(t)$ for all $t \geq 0$.*

Thus by setting-up a replicating portfolio, the writer of the perpetual American put is sure to always be able to afford to pay-off the buyer.

Note that in the European case a self-financing hedging portfolio is trivially replicating, as the price of European derivatives has been defined as the value of such portfolios.

However in the American case a replicating portfolio need not be self-financing: if the buyer does not exercise at an optimal exercise time, the writer must withdraw cash from the portfolio in order to replicate the derivative.

This leads to the definition of portfolio generating a cash flow.

Definition 6. *A portfolio* $\{h_S(t), h_B(t)\}_{t\geq 0}$ *with value* $\{V(t)\}_{t\geq 0}$ *is said to generate a* **cash flow** *with rate* $c(t)$ *if* $\{c(t)\}_{t\geq 0}$ *is adapted to* $\{\mathcal{F}_W(t)\}_{t\geq 0}$ *and*

$$dV(t) = h_S(t)dS(t) + h_B(t)dB(t) - c(t)dt \tag{4}$$

Remark 1. Note that the cash flow has been defined so that $c(t) > 0$ when the investor withdraws cash from the portfolio (causing a decrease of its value).

Theorem 5. *The portfolio given by*

$$h_S(t) = v_L'(S(t)), \quad h_B(t) = \frac{v_L(S(t)) - h_S(t)S(t)}{B(0)e^{rt}}$$

is replicating the perpetual American put while generating the case flow $c(t) = rK\mathbb{I}_{S(t)\leq L}$ *(i.e., cash is withdrawn at the rate* rK *whenever* $S(t) \leq L$*, provided of course the buyer does not exercise the derivative).*

Proof. By definition, $V(t) = h_S(t)S(t) + h_B(t)B(t) = v_L(S(t)) = \widehat{\Pi}(t)$, hence the portfolio is replicating. Moreover

$$dV(t) = d(v_L(S(t))) = h_S(t)dS(t) + \frac{1}{2}v_L''(S(t))\sigma^2 S(t)^2 dt. \tag{5}$$

Now, a straightforward calculation shows that $v_L(x)$ satisfies

$$-rv_L + rxv_L' + \frac{1}{2}\sigma x^2 v_L'' = -rK\mathbb{I}_{x\leq L},$$

a relation which was already used in step 1 in the proof of Theorem 2. It follows that

$$\begin{aligned}\frac{1}{2}v_L''(S(t))\sigma^2 S(t)^2 dt &= r(v_L(S(t)) - S(t)h_S(t))dt - rK\mathbb{I}_{S(t)\leq L}dt\\ &= h_B(t)dB(t) - rK\mathbb{I}_{S(t)\leq L}dt.\end{aligned}$$

Hence (5) reduces to (4) with $c(t) = rK\mathbb{I}_{S(t)\leq L}$, and the proof is complete. ☐

Remarks on American put options with finite maturity

The pricing function $v_L(x)$ of perpetual American puts satisfies

$$- rv_L + rxv_L' + \frac{1}{2}\sigma^2 x^2 v_L'' = 0 \quad \text{when } x > L, \tag{6}$$

$$v_L(x) = (K - x), \quad \text{for } x \leq L, \quad v_L'(L) = -1. \tag{7}$$

It can be shown that the pricing function of American put options with finite maturity satisfies a similar problem. Namely, letting $\widehat{P}(t)$ be the fair price at time t of the American put with strike K and maturity $T > t$, it can be shown that $\widehat{P}(t) = v(t, S(t))$, where $v(t, x)$ satisfies

$$\partial_t v + rx\partial_x v + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v = rv, \quad \text{if } x > x_*(t), \tag{8}$$

$$v(t, x) = (K - x), \quad \text{for } x \leq x_*(t), \quad \partial_x v(t, x_*(t)) = -1, \tag{9}$$

$$v(T, x) = (k - x)_+, \quad x_*(T) = K, \tag{10}$$

which is a **free-boundary value problem**. While a numerical solution of the previous problem can be found using the finite difference method, the price of the American put option is most commonly computed using binomial tree-approximations, see for instance [?].

MARTINGALE / SUPER MARTINGALE / SUBMARTINGALE

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<u>DEFINITION</u> LET $\{X(t)\}_{t\geq 0}$ BE A STOCHASTIC PROCESS SUCH THAT $X(t) \in L^1(\Omega)$ FOR ALL $t \geq 0$ AND $\{X(t)\}_{t\geq 0}$ IS ADAPTED TO $\{\mathcal{F}_W(t)\}_{t\geq 0}$. THEN

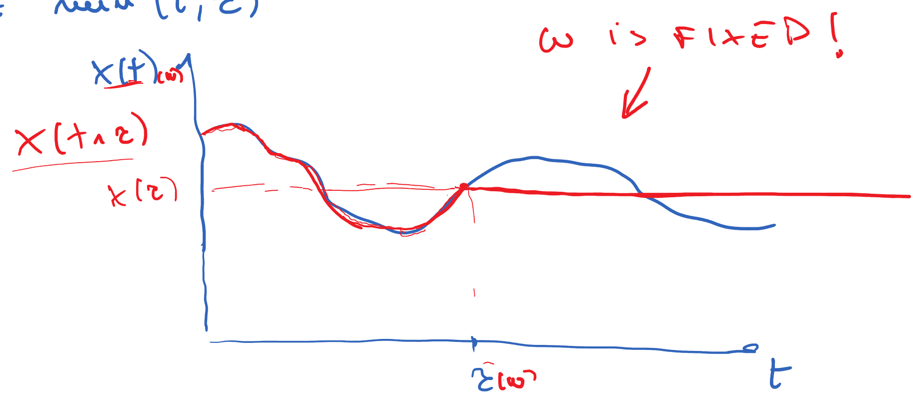
IF $\mathbb{E}[X(t) | \mathcal{F}_W(s)] = X(s)$ $\forall s \leq t$, THEN $\{X(t)\}_{t\geq 0}$ IS CALLED A MARTINGALE

IF $\mathbb{E}[X(t) | \mathcal{F}_W(s)] \leq X(s)$ $\forall s \leq t$, THEN $\{X(t)\}_{t\geq 0}$ IS CALLED A SUPERMARTINGALE

IF $\mathbb{E}[X(t) | \mathcal{F}_W(s)] \geq X(s)$ $\forall s \leq t$, THEN $\{X(t)\}_{t\geq 0}$ IS CALLED A SUBMARTINGALE

OPTIONAL SAMPLING THEOREM

$t \wedge \tau = \min(t, \tau)$



$\{X(t\wedge\tau)\}_{t\geq 0}$ IS THE STOCHASTIC PROCESS $\{X(t)\}_{t\geq 0}$ STOPPED AT TIME τ