den 5 mars 2021 16:16

$$\begin{cases} dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dW(t) \\ dB(t) = r(t)B(t)\,dt \end{cases}$$

1+1 DIMENSIONAL MARKET IN THE PHYSICAL PROBABILITY

IF $\sigma(t) > 0$, WE CAN DERIVE A (UNIQUE) EQUIVALENT RISK-NEUTRAL PROBABILITY, IN WHICH

$$dS(t) = r(t)S(t)\,dt + \sigma(t)S(t)\,d\tilde{W}(t)$$

IT FOLLOWS THAT THE DISCOUNTED STOCK-PRICE $S^*(t) = D(t)S(t)$,

WHERE $D(t) = \exp\left(-\int_0^t r(s)\,ds\right)$, IS

A $\tilde{\mathbb{P}}$-MARTINGALE RELATIVE TO $\{\mathcal{F}_W(t)\}_{t \geq 0}$

FROM HERE WE DERIVE THAT ANY SELF-FINANCING PORTFOLIO $\{h_S(t), h_B(t)\}_{t\geq 0}$ INVESTED IN THIS MARKET IS NOT AN ARBITRAGE.

$\Rightarrow$ 1+1 DIMENSIONAL MARKETS WITH $\sigma(t) > 0 \ \forall t$ ARE ARBITRAGE FREE.

MOREOVER, THE ARBITRAGE-FREE PRICE OF THE EUROPEAN DERIVATIVE WITH PAY-OFF Y AND MATURITY $T > 0$ IS GIVEN BY THE RISK-NEUTRAL PRICING FORMULA:

$$\Pi_Y(t) = \tilde{\mathbb{E}}\left[\frac{D(T)}{D(t)} Y \,\middle|\, \mathcal{F}_W(t)\right]$$

AND WE HAVE SHOWN THAT ANY EUROPEAN DERIVATIVE PRICED IN THIS WAY CAN BE REPLICATED BY SELF-FINANCING PORTFOLIOS $\{h_S(t), h_B(t)\}_{t\geq 0}$

$\Rightarrow$ 1+1 DIMENSIONAL MARKETS WITH $\sigma(t) > 0 \ \forall t$ ARE COMPLETE!

REMARK: N+1 DIMENSIONAL MARKETS ARE COMPLETE IF AND ONLY IF THE

REMARK: N+1 DIMENSIONAL MARKETS ARE COMPLETE IF AND ONLY IF THE VOLATILITY MATRIX IS NON-SINGULAR

IF NO FURTHER CONDITIONS ARE GIVEN ON THE MARKET PARAMETERS $\mu(t), \sigma(t), r(t)$, IT IS NOT POSSIBLE TO COMPUTE EXPLICITLY THE (RISK-NEUTRAL) PRICE OF THE DERIVATIVE.

MODELS:

• μ, σ, r CONSTANTS $\Rightarrow$ BLACK-SCHOLES MARKET

IN THIS CASE, WHEN $Y = g(S(T))$,

$$\Pi_Y(t) = v_g(t, S(t)), \quad h_S(t) = \partial_x v_g(t, S(t))$$

WHERE

$$v_g(t,x) = e^{-r\tau} \int_{\mathbb{R}} g\left(x e^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y}\right) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$\tau = T - t$

•• LOCAL VOLATILITY MODELS:

μ, r CONSTANT; $\sigma(t)S(t) = \beta(t, S(t))$

EXAMPLE: CEV MODEL, WHERE $\beta(t, S(t)) = \sigma_0 S(t)^\delta$

IN THIS CASE v_g SATISFIES THE PDE

$$\begin{cases} \partial_t v_g + rx\partial_x v_g + \frac{1}{2}\sigma^2 x^{2\delta} \partial_x^2 v_g = r v_g \\ v_g(T,x) = g(x) \end{cases}$$

IN THIS CASE THE GENERAL SOLUTION IS NOT KNOWN, BUT A NUMERICAL SOLUTION CAN BE FOUND E.G. WITH THE FINITE DIFFERENCE METHOD.

••• STOCHASTIC VOLATILITY MODELS

μ, r CONSTANT, $\sigma(t)$ SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION

EXAMPLE: HESTON MODEL, WHERE

$$d\sigma^2(t) = a(b - \sigma^2(t))dt + c\sigma(t)\, d\tilde{W}(t)$$

(IN THE RISK-NEUTRAL PROBABILITY)

IN THIS CASE $\Pi_Y(t) = N_g(t, S(t), \sigma^2(t))$

WHERE N_g SATISFIES A PDE IN 1+2 DIMENSIONS.

REMARK: VOLATILITY AND STOCHASTIC VOLATILITY MODELS ARE IMPORTANT TO REPRODUCE VOLATILITY CURVES IN THE MARKET

REMARK: AMERICAN DERIVATIVES ARE MUCH MORE COMPLICATED TO STUDY AND EVEN IN BLACK-SCHOLES MARKET NO EXPLICIT FORMULA SOLUTION FOR EVEN THE SIMPLEST DERIVATIVE IS KNOWN. (EXCEPT FOR THE PERPETUAL AMERICAN PUT).

MOREOVER, IF $r > 0$ AND THE STOCK PAYS NO DIVIDEND, THEN THE AMERICAN AND THE EUROPEAN CALL HAVE THE SAME VALUE, BECAUSE IT IS NEVER OPTIMAL TO EXERCISE THE AMERICAN CALL BEFORE MATURITY.

WE ALSO DISCUSSED FORWARDS AND FUTURES (THE STUDY OF FORWARDS LEADS US TO THE CONCEPT OF FORWARD PROBABILITY, IN WHICH THE FORWARD PRICE OF THE ASSET IS A MARTINGALE) (SEC. 6.8)

AND WE ALSO STUDIED SOME INTEREST RATE CONTRACT, IN PARTICULAR ZERO-COUPON AND COUPON BONDS AND INTEREST RATE SWAPS. THE VALUATION OF ALL THESE CONTRACTS DEPEND ON THE VALUE OF THE ZCB, $B(t,T)$.

THE VALUE OF ANY OTHER INTEREST RATE CONTRACT IS A FUNCTION OF $B(t,T)$ (WITH DIFFERENT MATURITIES)

VALUATION OF ZCB'S:

METHOD 1 : CLASSICAL APPROACH

METHOD 2 : HJM APPROACH

CLASSICAL APPROACH

$$B(t,T) = \tilde{E}\left[e^{-\int_t^T r(s)\,ds} \,\middle|\, \mathcal{F}_W(t) \right]$$

is computed by solving a SDE for $r(t)$ in the risk-neutral probability

EXAMPLE (EXERCISE 6.35)

$$dr(t) = a(t)(b(t)-r(t))dt + c(t)\sqrt{r(t)+\delta(t)}\, d\tilde{W}(t)$$

ASSUME $B(t,T) = v(t, r(t))$, DERIVE THE PDE SATISFIED BY v.

$$d\big(D(t)v(t,r(t))\big) = \big(dD(t)\big) v(t,r(t))$$

$$+ \; D(t)\, dv(t,r(t)) \; + \; \underbrace{dD\, dv}_{0}$$

(BECAUSE $dD(t) = -r(t)D(t)dt$)

$$= -r(t)D(t)v(t,r(t))dt + D(t)\big(\partial_t v\, dt + \partial_x v\, dr(t)$$

$$+ \; \tfrac{1}{2}\partial_x^2 v \; dr(t)dr(t)\big)$$

$$= -r(t)D(t)v\,dt + D(t)\big[\partial_t v\, dt + \partial_x v\; a(t)(b(t)-r(t))dt$$

$$+ \; c(t)\sqrt{r(t)+\delta(t)}\; d\tilde{W}(t) + \tfrac{1}{2}\partial_x^2 v \;\; c(t)^2 (r(t)+\delta(t))\, dt\big]$$

$$= D(t)\left[-r(t)v + \partial_t v + a(t)(b(t)-r(t))\partial_x v + \tfrac{1}{2}c(t)^2(r(t)+\delta(t))\partial_x^2 v\right]dt$$

$$+ \quad (\,---\,)\, d\tilde{W}(t)$$

HENCE, IF v SATISFIES

$$(*)\ \partial_t v + a(t)(b(t)-x)\partial_x v + \frac{1}{2}c(t)^2(x+\delta(t))\partial_x^2 v = x v$$

THEN $D(t)\,v(t, r(t))$ IS A $\tilde{P}$-MARTINGALE:

$$\tilde{E}\left[D(\tau)\, v(\tau, r(\tau)) \mid \mathcal{F}_W(t)\right] = \underline{D(t)\, v(t, r(t))}$$

REPLACING $\tau = T$ AND IMPOSING THE TERMINAL CONDITION $v(T, x) = 1$, WE OBTAIN

$$v(t, r(t)) = \frac{1}{D(t)}\tilde{E}\left[D(T)\,1 \mid \mathcal{F}_W(t)\right]$$

$$= \underline{B(t,T)}$$

THE SOLUTION OF $(*)$ WITH TERMINAL VALUE $v(T,x) = 1$ CAN BE FOUND USING THE ANSATS

$$v(t,x) = e^{-C(t)x - A(t)} \quad \text{(AFFINE MODEL)}$$

REMARK $(*)$ REDUCES TO NON-LINEAR ODE'S ON $C(t), A(t)$ WHICH CAN BE EXPLICITELY SOLVED FOR SOME SPECIAL $a(t), b(t), c(t), \delta(t)$ (E.G., THE CIR MODEL)