Ö5_V3_14april

den 14 april 2021 15:02

HG:3 3. $F: \mathbb{P}_n \to \mathbb{P}_n$ $\qquad \mathbb{P}_n = \{ p(t) = a_0 t \ldots + a_{n-1}t^{n-1} + a_n t^n \}$, $a_i \in \mathbb{R}$

$$u \to F(u) = \underbrace{\frac{d^2u}{dt^2} - 2t\frac{du}{dt}}$$

a) F är linjär avbildning

Lösning: $p, q \in \mathbb{P}_n$, $\alpha, \beta \in \mathbb{R}$

$$\alpha p + \beta q \in \mathbb{P}_n$$

$$F(\underbrace{\alpha p+\beta q}) = \frac{d^2}{dt^2}(\alpha p+\beta q) - 2t\frac{d}{dt}(\alpha p+\beta q) =$$

$$= \alpha\frac{d^2}{dt^2}p + \beta\frac{d^2q}{dt^2} - 2t\left[\alpha\frac{dp}{dt} + \beta\frac{dq}{dt}\right] =$$

$$= \alpha\left[\frac{d^2p}{dt^2} - 2t\frac{dp}{dt}\right] + \beta\underbrace{\left[\frac{d^2p}{dt^2} - 2t\frac{dq}{dt}\right]} =$$

$$= \alpha F(p) + \beta F(q)$$

$\square$

b) $\mathbb{P}_3$ $\quad B = \{\underbrace{1, t, t^2, t^3}\}$? Matrisen av F på $\mathbb{P}_3$.

$$F: \mathbb{P}_3 \to \mathbb{P}_3 \quad (B \text{ på båda})$$

$$p(t) = a_0 1 + a_1 t + a_2 t^2 + a_3 t^3 \qquad a_0, a_1, a_2, a_3 \in \mathbb{R}.$$

$$F(p(t)) = F(a_0 1 + a_1 t + a_2 t^2 + a_3 t^3) \overset{\text{linjär}}{=}$$

$$= a_0\underbrace{F(1)}_{v_1} + a_1\underbrace{F(t)}_{v_2} + a_2\underbrace{F(t^2)}_{v_3} + a_3\underbrace{F(t^3)}_{v_4}$$

$$= \begin{bmatrix} F(1) & F(t) & F(t^2) & F(t^3) \\ & & & \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}}_{[F]}$$

$$F(1) = \frac{d^2}{dt^2}(1) - 2t\frac{d}{dt}(1) = 0 - 2t.0 = 0 = 0.1 + 0.t + 0t^2 + 0t^3$$

$$0 = (0,0,0,0)_B$$

$$F(t) = \frac{d^2(t)}{dt^2} - 2t.\frac{d}{dt}(t) = 0 - 2t.1 = -2t = (0,-2,0,0)$$

$$0.1 - 2t + 0t^2 + 0t^3$$

$$F(t^2) = \frac{d^2}{dt^2}(t^2) - 2t\frac{d}{dt}(t^2) = 2.1 - 2t(2t) = \underbrace{2 - 4t^2}$$

$$= 2.1 + 0.t - 4t^2 + 0t^3 = (2,0,-4,0)_B$$

$$F(t^3) = \frac{d^2}{dt^2}(t^3) - 2t\frac{d}{dt}(t^3) = 3.2t - 2t\,3t^2 = \underbrace{6t - 6t^3}$$

$$= 0.1 + 6t + 0t^2 - 6t^3 = (0,6,0,-6)_B$$

HG 4. 12 $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$

$p(t) \longrightarrow T(p(t)) = t \cdot p'(t+1) + p(t)$

a) T är linjär avbildning

b) $[T]_B$ $B = \{1, t, t^2\}$ standard bas.

c) egenvärden / egenvektorer till $[T.]$ $(Tp = \lambda p)$

Lösning. $p, q \in \mathbb{P}_2$, $\alpha, \beta \in \mathbb{R}$ $\boxed{(t+1) = s \quad dt = ds}$

a)

$\alpha p + \beta q$

$$T(\alpha p + \beta q) = t\,(\alpha p + \beta q)'(t+1) + (\alpha p + \beta q)(t)$$

$$= t \cdot \frac{d}{ds}(\alpha p + \beta q)(s) + \alpha p(t) + \beta q(t)$$

$$= t\alpha \frac{d}{ds} p(s) + t\beta \frac{d}{ds} q(s) + \alpha p(t) + \beta (q(t))$$

$$= t\alpha \frac{d}{dt} p(t+1) + t\beta \frac{d}{dt} q(t+1) + \alpha p(t) + \beta q(t)$$

$$= \alpha \left(t \frac{d}{dt} p(t+1) + p(t) \right) + \beta \left(t \frac{d}{dt} q(t+1) + q(t) \right)$$

$$= \alpha T(p) + \beta T(q), \quad \forall p, q \in \mathbb{P}_2, \ \forall \alpha, \beta \in \mathbb{R}$$

Då är T linjär avbildning.

$\mathbb{P}_2$, $B = \{1, t, t^2\}$ $p(t) \in \mathbb{P}_2$

$p(t) = a_0 1 + a_1 t + a_2 t^2 \Rightarrow$

$$T(p(t)) = T(a_0 1 + \dots + a_2 t^2) = a_0 T(1) + a_1 T(t) + a_2 T(t^2)$$

$$= \begin{bmatrix} T(1) & T(t) & T(t^2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$T(1) = t \frac{d}{dt}(1) + 1 = 0 + 1 = 1 = (1, 0, 0)_B$$

$T(1) = t\frac{d}{dt}(1) + 1 = 0 + 1 = 1 = (1,0,0)_B$

$T(t) = t\frac{d}{dt}(t+1) + t = t.1 + t = 2t = (0,2,0)_B$

$T(t^2) = t\frac{d}{dt}\left[(t+1)^2\right] + t^2 = t\frac{d}{dt}\left(t^2+2t+1\right) + t^2 =$

$= t(2t+2) + t^2 = 2t^2 + 2t + t^2 =$

$= 3t^2 + 2t = (0,2,3)_B$

$$[T]_{B \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

B standard

c) $\begin{cases} \text{egenvärden} \\ \text{egenvektorer} \end{cases}$ $\boxed{Tv = \lambda v}$ $\boxed{Tp = \lambda p}$

Karakteristik ekv: $\det\left([T] - \lambda I\right) = 0$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0 \qquad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$(1-\lambda)=0 \Rightarrow \lambda_1 = 1$

$(2-\lambda)=0 \Rightarrow \lambda_2 = 2$

$(3-\lambda)=0 \Rightarrow \lambda_3 = 3$

$\lambda_3 = 3 \quad v_3 = \text{egenvektor}$

$$\begin{bmatrix} 1-3 & 0 & 0 \\ 0 & 2-3 & 2 \\ 0 & 0 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -2x = 0 & \Rightarrow \boxed{x = 0} \\ -y + 2z = 0 & \Rightarrow \boxed{y = 2z} \end{cases} \qquad \begin{aligned} x &= 0 \\ y &= 2\alpha \\ z &= \alpha \end{aligned} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2\alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\alpha \in \mathbb{R}$

$\lambda_3 = 3, \quad v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$$\lambda_2 = 2 \qquad \begin{bmatrix} 1-2 & 0 & 0 \\ 0 & 2-2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 2$ $\quad \begin{bmatrix} 1-2 & 0 & 0 \\ 0 & 2-2 & 2 \\ 0 & 0 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{gauseli}} \begin{bmatrix} +1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} 1x = 0 \\ 1z = 0 \\ y = \text{fri} \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}.$

$\lambda_2 = 2 \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 1$ $\quad v_1$ lösning till H. syst.

$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 2-1 & 2 \\ 0 & 0 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} 0 = 0 & \quad x \text{ fri} \\ y + 2z = 0 \\ 2z = 0 & \Rightarrow \boxed{z = 0} \end{cases} \qquad y + 2(0) = 0 \Rightarrow \boxed{y = 0}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \ \alpha \in \mathbb{R}$

$\lambda_3 = 3 \longrightarrow v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2 \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Bas med egenvektorer

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\underbrace{}_{v_1} \quad \underbrace{}_{v_2} \quad \underbrace{}_{v_3}$

$v_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0.1 + 2t + 1t^2 \Rightarrow P_3(t) = t^2 + 2t$

$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0.1 + 1t + 0t^2 \Rightarrow P_2(t) = t$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1.1 + 0t + 0t^2 \Rightarrow P_3(t) = 1$

_____, ________, _________, ________

Kapitel 2:

1) Bestäm $\cos\theta$, där $\theta = \measuredangle(p,q)$ $\quad p(t) = 3t+1 \in \mathbb{P}_2$, $q(t) = 5t^2+3 \in \mathbb{P}_2$

$\mathbb{P}_2$ med skalärprodukten $\langle\cdot,\cdot\rangle$

$$\langle f,g\rangle = \int_{-1}^{1} f(t)\,g(t)\,dt$$

Lösning. $\cos\theta = \dfrac{\langle p,q\rangle}{\|p\|\cdot\|q\|}$ $\quad \|p\|^2 = \langle p,p\rangle \Rightarrow \|p\| = \sqrt{\langle p,p\rangle}$

$\|q\|^2 = \langle q,q\rangle \quad \|q\| = \sqrt{\langle q,q\rangle}$

$p(t) = 3t+1$

$$\|p\|^2 = \int_{-1}^{1} p(t)\cdot p(t)\,dt = \int_{-1}^{1}(3t+1)^2\,dt = \int_{-1}^{1}(9t^2+6t+1)\,dt =$$

$$\left[\cancel{9}\cdot\frac{1}{\cancel{3}}t^3 + \cancel{6}\cdot\frac{1}{\cancel{2}}t^2 + t\right]_{-1}^{1} = \left[3t^3 + 3t^2 + t\right]_{-1}^{1} =$$

$$(3 + \cancel{3} + 1) - (-3 + \cancel{3} - 1) = 6 + 2 = 8$$

$$\|p\| = \sqrt{8}$$

$$\|q\|^2 = \langle q,q\rangle = \int_{-1}^{1}(5t^2+3)^2\,dt = \int_{-1}^{1}(25t^4+30t^2+9)\,dt =$$

$$\left[25\,\frac{1}{5}t^4 + 30\cdot\frac{1}{3}t^2 + 9t\right]_{-1}^{1} = \left[5t^5+10t^3+9t\right]_{-1}^{1} =$$

$$(5+10+9) + (5+10+9) = 48$$

$$\|q\| = \sqrt{48}$$

$$\langle p,q\rangle = \int_{-1}^{1}(3t+1)(5t^2+3)\,dt = \int_{-1}^{1}(15t^3+9t+5t^2+3)\,dt$$

$$\left[\cancel{15\frac{t^4}{4}} + 5\,\frac{t^3}{3} + \cancel{9\,\frac{t^2}{2}} + 3t\right]_{t=-1}^{t=1} =$$

$$\left(\frac{5}{3} + 3\right) + \left(\frac{5}{3}+3\right) = \frac{10}{3} + 6 = \frac{28}{3}$$

$$\cos\theta = \frac{\langle p,q\rangle}{\|p\|\,\|q\|} = \frac{28}{3\cdot\sqrt{8}}\,\frac{1}{\sqrt{48}} = \dots = \frac{7}{6\sqrt{6}}$$

$$\cos\theta = \frac{\dots}{\|p\|\,\|q\|} = \frac{\alpha\cdot}{3\cdot\sqrt{8}}\;\frac{1}{\sqrt{48}} = \dots = \frac{1}{6\sqrt{6}}$$

HG: K2 Upp 4 $x=(x_1,x_2)$ $y=(y_1,y_2)$ $\in \mathbb{R}^2 = V,\ \langle\cdot,\cdot\rangle$

$$\langle x,y\rangle = 2x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

a) $\langle\cdot,\cdot\rangle$ är skalärprodukt

b) Bestäm en ON-bas för $\mathbb{R}^2$ med $\langle\cdot,\cdot\rangle$.

Lösning:

a) $\langle u,v\rangle = \langle v,u\rangle$

b) $\langle \alpha u,v\rangle = \alpha\langle u,v\rangle$

c) $\langle u+w,v\rangle = \langle u,v\rangle + \langle w,v\rangle$

$\longrightarrow$ d) $\langle u,u\rangle \geq 0$, $\boxed{\langle u,u\rangle = 0 \Leftrightarrow u=0}$

d) $\langle x,x\rangle = 2x_1^2 - \underbrace{2x_1x_2 - 2x_2x_1}\, + 5x_2^2 =$

$= 2x_1^2 - 4x_1x_2 + 5x_2^2$

$= \underbrace{2x_1^2 - 4x_1x_2 + 2x_2^2} + 3x_2^2$

$= 2(\underbrace{x_1^2 - 2x_1x_2} + x_2^2) + 3x_2^2$

$2(x_1-x_2)^2 + 3x_2^2 \geq 0$

$$\langle x,x\rangle = 2\underbrace{(x_1-x_2)^2}_{\geq 0} + \underbrace{3x_2^2}_{\geq} = 0 \Longrightarrow x=(x_1,x_2)=(0,0)$$

$$\begin{cases} 3x_2^2 = 0 \Leftrightarrow \boxed{x_2=0} \\ 2(x_1-x_2)^2 = 0 \Leftrightarrow (x_1-x_2)^2 = 0 \Leftrightarrow (x_1)^2 = 0 \Leftrightarrow \boxed{x_1=0} \end{cases}$$

Då är $\langle\cdot,\cdot\rangle$ skalär produkt.

b) $B = \{\bar{e}_1, \bar{e}_2\}$ standard bas när $(\cdot,\cdot)$ standa skalär produkt.

$\bar{e}_1 = (1\ 0)$ $\rightarrow$ $\|\bar{e}_1\| = \sqrt{\langle e_1, e_1\rangle}$ $\langle x,x\rangle = 2(x_1-x_2)^2 + 3x_2^2$

$\bar{e}_1 = (1,0) \quad \to \quad \|\bar{e}_1\|_{\langle\cdot,\cdot\rangle} = \sqrt{\langle e_1, e_1 \rangle} \qquad \underline{\langle x, x\rangle = 2(x_1 - x_2)^2 + 3x_2^2}$

$\|e_1\|^2 = \langle e_1, e_1 \rangle = 2(1-0)^2 + 3(0)^2 = 2$

$\|e_1\| = \sqrt{2}$

$\|e_2\|^2 = \langle e_2, e_2 \rangle = 2(0-1)^2 + 3(1)^2 = 2 + 3 = 5$

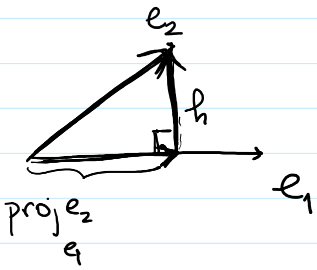
$e_2 = (0,1) \quad \|e_2\| = \sqrt{5}$

$\langle e_1, e_2 \rangle = 2 x_1 y_1 - 2 x_1 y_2 - 2 x_2 y_1 + 5 x_2 y_2 =$

$e_1 = (x_1, x_2) = (1,0) \qquad = 2.1.0 - 2.1.1 - 2\cancel{0}.1 - 5\cancel{0}.1 =$

$e_2 = (y_1, y_2) = (0,1) \qquad = -2$

Mål: ON-bas med $\langle \cdot, \cdot \rangle$



$\text{proj}_{e_1} e_2 + h = e_2$

$h = e_2 - \text{proj}_{e_1} e_2 \qquad \underline{(G.S)}$

$\text{proj}_{e_1} e_2 = \dfrac{\langle e_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1$

$\langle x, y \rangle = 2x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$

$(x_1, x_2) = (0,1)$

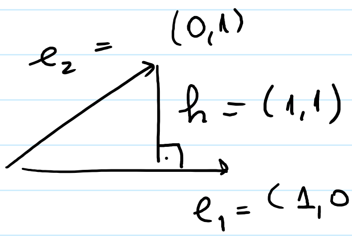
$(y_1, y_2) = (1,0)$

$\langle x, y \rangle = 2.\cancel{1}.0 - 2\cancel{0}.1 - 2.1.1 + 5.\cancel{0}.1$

$h = (0,1) - \dfrac{(\cancel{-2})}{\cancel{2}} e_1$

$h = (0,1) + (1,0)$

$\boxed{h = (1,1)}$



$, \langle \cdot, \cdot \rangle$

O-Bas $\{e_1, h\}$ $\qquad$ ON-bas $\left\{ \dfrac{e_1}{\|e_1\|}, \dfrac{h}{\|h\|} \right\}$

$ON = \left\{ \frac{1}{\sqrt{2}}(1,0) \;,\; \frac{1}{\sqrt{3}}(1,1) \right\}$

$\|h\|^2 = \langle h, h \rangle = 2(x_1 - x_2)^2 + 3x_2^2 = 2(\cancel{1-1})^2 + 3(1)^2 = 3$

$\|h\| = \sqrt{3}$

K2 Upp 6 $\quad A \in \mathbb{R}^{n \times n}$ är inverterbar $(A^{-1} \checkmark)$

$u, v \in \mathbb{R}^n \qquad \langle u, v \rangle = \underbrace{(Au) \cdot (Av)} = \underbrace{(Au)^T (Av)}$

Bevisa att $\langle u, v \rangle$ är skalär produkt

a) $\langle u, v \rangle = \langle v, u \rangle$

b) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

c) $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$

d) $\langle u, u \rangle \geq 0 \quad , \quad \langle u, u \rangle = 0 \iff u = 0$

$\langle u, u \rangle = (Au) \cdot (Au) = (Au)^T \cdot (Au)$

$Au = b \in \mathbb{R}^n \qquad A_{n \times n} \qquad b^T \cdot b$

$(Au)^T = b^T$

$[b_1\, b_2 \dots b_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = (b_1)^2 + \dots + (b_n)^2$

$(Au)_1^2 + (Au)_2^2 + \dots (Au)_n^2 \geq 0$

$(Au)_1^2 + \dots (Au)_n^2 = 0 \iff \begin{matrix} (Au)_1 = 0 \\ \vdots \\ (Au)_n = 0 \end{matrix} \Big\} \; \boxed{Au = 0}$

$b_1^2 + b_2^2 + \dots + b_n^2 = 0 \qquad \boxed{u = A^{-1} \cdot 0}$

$\begin{matrix} b_1 = 0 \\ b_2 = 0 \\ \vdots \\ b_n = 0 \end{matrix} \rightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} u \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff \boxed{u = 0}$

$u = 0$ endast lösning.

därför att

A är inverterbar $\square$