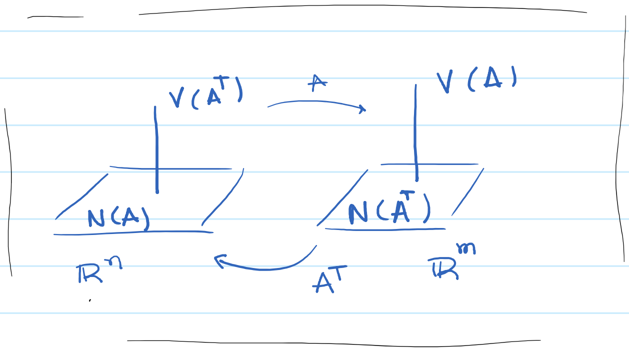
L6_V3_15april

den 14 april 2021 20:00

HG (2.3)

$V = \mathbb{R}^n$ med $\langle x, y \rangle = x^T y$, $\forall x, y \in \mathbb{R}^n$.



Sats: För $A = [a_1\ a_2 \dots a_n] \in \mathbb{R}^{m\times n}$

är $[V(A)]^\perp = N(A^T)$

Bevis:

$x \in [V(A)]^\perp \iff \langle a_i, x \rangle = 0$ för varje kolonn a_i av A. $(i = 1, 2, \dots, n)$

$\iff a_i^T \cdot x = 0 \quad i = 1, \dots n$

$\iff A^T \cdot x = 0$ $\quad x$ är lösning till Homogena system

$\iff x \in N(A^T)$ $\square$

Mål: Titta igen några uppgifter i kapitel 1 som behöver detta begreppen.

Kapitel 1 Upp. 33 : Bestäm matris A så att de givna mängderna utgör nollrummet $N(A)$

a) $M = \left\{ \begin{pmatrix} t \\ s \\ s-t \end{pmatrix} : s, t \in \mathbb{R} \right\} \Rightarrow \begin{array}{l} x = t \\ y = s \\ z = s-t \end{array} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_1} + s \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_2}$

$M = \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{v_2} \right\}$

Fråga: Vilken A har $x = v_1$ och $x = v_2$ som ortogonala vektorer till sina råder ???

$\langle a_i, x \rangle = 0$

$a_i^T \cdot x = 0$

Om vi tar $A^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{l} - v_1 - \\ - v_2 - \end{array}$

$A^T \cdot x = 0 \iff \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Då $\begin{cases} 1x + 0y - z = 0 \\ 1y + z = 0 \end{cases} \Rightarrow \begin{array}{l} x = z \\ y = -z \\ z = z \end{array} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Då $\begin{cases} 1x + 0y - z = 0 \\ \quad\; 1y + z = 0 \end{cases} \Rightarrow \begin{aligned} x &= z \\ y &= -z \\ z &= z \end{aligned} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$

Då $\boxed{A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}}$ och $\underline{N(A) = M = \mathrm{Span}\{v_1, v_2\}}$

efter som $(1,-1,1) \cdot v_1 = (1\ -1\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1\ 0\ -1 = \underline{0}$

$(1,-1,1) \cdot v_2 = (1\ -1\ 1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \underline{0\ -1+1 = 0}$

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33 b) $M = \left\{ \begin{pmatrix} t \\ t+u \\ -s \\ s-u \end{pmatrix} ; t, u, s \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + u \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}_{v_2} + s \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{v_3} \right\}$

$M = N(A)$ $\quad A = ?$

$A^T = \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

gausselimination $\downarrow$

$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} y_1 + y_4 &= 0 \\ y_2 - y_4 &= 0 \\ y_3 - y_4 &= 0 \\ y_4 &= \alpha \end{aligned} \Rightarrow \begin{aligned} y_1 &= -y_4 \\ y_2 &= y_4 \\ y_3 &= y_4 \\ y_4 &= \alpha \end{aligned}$

$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}}_{a_1} ; \alpha \in \mathbb{R}$

$\langle a_1, v_1 \rangle = (1,-1,-1,-1)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0$

$\langle a_1, v_2 \rangle = (1,-1,-1,-1)\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 0$

$\langle a_1, v_3 \rangle = (1,-1,-1,-1)\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = 0$

$$\langle a_1, v_3\rangle = (1,-1,-1,-1)\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = 0$$

Då $A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$ så att $N(A) = M$

Kapitel 1

36 $U \subseteq \mathbb{R}^5$ $\quad U = \text{Span}\left\{ \underbrace{\begin{pmatrix}1\\2\\1\\2\\1\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}1\\3\\1\\1\\2\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}2\\1\\4\\3\\-3\end{pmatrix}}_{u_3}, \underbrace{\begin{pmatrix}3\\2\\4\\8\\-2\end{pmatrix}}_{u_4} \right\}$

a) Bestäm $\dim U$

b) Ange A sådan att $N(A) = U$

Lösning

1) $U = \text{Span}\{u_1, u_2, u_3, u_4\}$

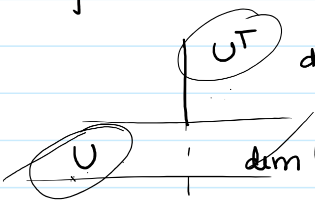
$$\begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} \xrightarrow{\text{Gauss elimination}} \begin{pmatrix} 1 & 0 & 5 & 7 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

då $\quad u_4 = \frac{9}{2}u_1 - \frac{5}{2}u_2 + \frac{1}{2}u_3$

$$U = \text{Span}\{u_1, u_2, u_3, u_4\} = \text{Span}\{u_1, u_2, u_3\}$$

$B = \{u_1, u_2, u_3\}$ Linjärt Oberoende $\Rightarrow$ B bas för U $\Rightarrow$ $\dim U = 3$

$U \subseteq \mathbb{R}^5$



$\dim U^T = 2$

$\dim U = 3$

$$\dim U + \dim U^T = 5$$

2) $A = ?$ Så att $N(A) = U$ $\qquad \langle a_i, x\rangle = 0 \qquad a_i^T \cdot x = 0$

$A^T \cdot x = 0$

$x \in U$!

$$A^T = \begin{array}{c} u_1^- \\ u_2^- \\ u_3^- \end{array}\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 & -3 \end{pmatrix} \xrightarrow{\text{Gausselim}} \begin{pmatrix} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & -1 \end{pmatrix}$$

$$A^T x = 0 \Rightarrow \begin{cases} x_1 = -6x_4 \\ x_2 = +1x_4 - 1x_5 \\ x_3 = +2x_4 + 1x_5 \\ x_4 = \alpha \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \alpha\begin{pmatrix} -6 \\ 1 \\ 2 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 = +2x_4 + 1x_5 \\ x_4 = \alpha \\ x_5 = \beta \end{cases} \Rightarrow \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \alpha \begin{pmatrix} \vdots \\ 2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \vdots \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -6 & 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = 0$$

$$(-6, 1, 2, 1, 0) \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = -6 + 2 + 2 + 2 + 0 = 0$$

$$(-6, 1, 2, 1, 0) \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} = -6 + 3 + 2 + 1 + 0 = 0$$

$$\boxed{N(A) = \text{Span}\{u_1, u_2, u_3\} = U}$$

☒

(*) Resultat.

$$\boxed{\text{proj}_U w = \text{proj}_{u_1} w + \text{proj}_{u_2} w + \dots + \text{proj}_{u_n} w}$$

$$\begin{cases} U = \text{span}\{u_1, u_2, \dots u_n\} \\ \quad \text{och} \\ \{u_1, u_2, \dots u_n\} \text{ ON-bas för } U \end{cases}$$