Ö7_V4_20april

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ELW Kapitel 17 : 1701a, 1704c, 1706f, 1707a

1701a. $\{a_n\}_{n=1}^{\infty}$ undersök om $\{a_n\}$ konvergent $a_n \to a$ / divergent.

a) $$a_n = \frac{4n^2+n+4}{(1+n)(2-3n)} = \frac{4n^2+n+4}{2-3n+2n-3n^2} = \frac{4n^2+n+4}{-3n^2-n+2}$$

$$a_n = \frac{\cancel{n^2}\left(4+\frac{1}{n}+\frac{4}{n^2}\right)}{\cancel{n^2}\left(-3-\frac{1}{n}+\frac{2}{n^2}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(4+\underbrace{\frac{1}{n}}_{\to 0}+\underbrace{\frac{4}{n^2}}_{\to 0}\right)}{\left(-3-\underbrace{\frac{1}{n}}_{\to 0}+\underbrace{\frac{2}{n^2}}_{\to 0}\right)} = -\frac{4}{3} = a$$

gränsvärde

Då $\{a_n\}_{n=1}^{\infty}$ är konvergent och $L = -\frac{4}{3}$ är gränsvärde.

1704c : $\{a_n\}_{n\in\mathbb{N}}$ konvergent, $L = ?$ eller divergent?

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Lösning:

$$a_n = \underbrace{\frac{1}{\sqrt{n^2+1}}}_{\text{max värde i summan}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \dots + \underbrace{\frac{1}{\sqrt{n^2+n}}}_{\text{minsta värde i summa}}$$

$$a_n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$$

$$\sqrt{n^2\left(1+\frac{1}{n^2}\right)}$$

$$a_n \leq \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}} \Longrightarrow a_n \leq \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$a_n \geq \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}$$

$$a_n \geq \frac{\cancel{n}}{}$$

$\cancel{n}\sqrt{1+1/n}$

$$\frac{1}{\sqrt{1+\frac{1}{n}}} \leq a_n \leq \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$n \to \infty$: $\frac{1}{\sqrt{1+\frac{1}{n}}} \to 1$, $\quad \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$

Enligt Squeeze Theorem $a_n \to 1$

$$\lim_{n\to\infty} a_n = 1 \qquad \{a_n\}_{n=1}^{\infty} \text{ konvergent.}$$

1706 f) Bestäm lösning y_n, $n \geq 0$

$$2y_{n+1} + y_n = 3 \qquad \boxed{y_0 = 2}$$

Lösning: $y_{n+1} + \frac{1}{2} y_n = \frac{3}{2}$

$$y_{n+1} = \underbrace{-\frac{1}{2} y_n + \frac{3}{2}}$$

$$y_0 = 2$$

$$y_1 = \underbrace{-\frac{1}{2} y_0 + \frac{3}{2}}$$

$$y_2 = -\frac{1}{2}(y_1) + \frac{3}{2} = \left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right) y_0 + \frac{3}{2}\right] + \frac{3}{2} =$$

$$y_3 = \underbrace{\left(-\frac{1}{2}\right)^{2} y_0 + \left(-\frac{1}{2}\right)^{1} \frac{3}{2} + \frac{3}{2}\left(-\frac{1}{2}\right)^{0}}$$

$$y_4 = \left(-\frac{1}{2}\right) y_3 + \frac{3}{2}$$

$$y_4 = \left(-\frac{1}{2}\right)^{3} y_0 + \underbrace{\left(-\frac{1}{2}\right)^{2} \frac{3}{2} + \frac{3}{2}\left(-\frac{1}{2}\right)^{1} + \frac{3}{2}\left(-\frac{1}{2}\right)^{0}}$$

$$y_n = \left(-\frac{1}{2}\right)^{n-1} y_0 + \underbrace{\sum_{k=0}^{n-2} \frac{3}{2} \cdot \left(-\frac{1}{2}\right)^{k}}$$

$$y_n = \left(-\frac{1}{2}\right)^{n-1} \cdot y_0 + \left(\frac{3}{2}\right) \sum_{k=0}^{n-2} \left(-\frac{1}{2}\right)^{k} \qquad q = \left(-\frac{1}{2}\right)$$

$$y_n = \left(-\tfrac{1}{2}\right)^{n-1} \cdot y_0 + \frac{3}{2} \sum_{k=0}^{n-2} \left(-\tfrac{1}{2}\right)^k \qquad q = \left(-\tfrac{1}{2}\right)$$

$$S_{n-2} = \sum_{k=0}^{n-2} q^k = q^0 + \cancel{q^1} + \dots + \cancel{q^{n-2}}$$

$$q S_{n-2} = (\cancel{q^1} + q^2 + \dots + \cancel{q^{n-1}})$$

$$(1-q) S_{n-2} = S_{n-2} - q S_{n-2} = q^0 - q^{n-1}$$

$$S_{n-2} = \frac{q^0 - q^{n-1}}{(1-q)} \quad \leftarrow$$

$$y_n = \left(-\tfrac{1}{2}\right)^{n-1} y_0 + \frac{3}{2} \frac{\left(1 - \left(-\tfrac{1}{2}\right)^{n-1}\right)}{\left(1 + \left(+\tfrac{1}{2}\right)\right)}$$

$$y_n = \left(-\tfrac{1}{2}\right)^{n-1} y_0 + \cancel{\tfrac{3}{2}} \frac{\left(1 - \left(-\tfrac{1}{2}\right)^{n-1}\right)}{\cancel{\left(\tfrac{3}{2}\right)}}$$

$$y_n = \left(-\tfrac{1}{2}\right)^{n-1} y_0 + 1 - \left(-\tfrac{1}{2}\right)^{n-1}$$

$$y_n = \left(-\tfrac{1}{2}\right)^{n-1} (y_0 - 1) + 1$$

$$y_n = \underbrace{\left(-\tfrac{1}{2}\right)^{n-1} (\underset{\substack{\uparrow \\ y_0}}{2} - 1)} + 1$$

$$\boxed{y_n = \left(-\tfrac{1}{2}\right)^{n-1} + 1} \quad \forall n \in \mathbb{N}.$$

1707 a) $y_{n+1} - y_n = e^n \qquad n \geq 1 \qquad \boxed{y_1 = 1}$

Lösung:

$$y_{n+1} = y_n + e^n$$

$$y_1 = 1$$

$y_2 = y_1 + e^1$

$y_3 = y_2 + e^2 = y_1 + e^1 + e^2$ $\qquad$ $y_4 = y_1 + e^1 + e^2 + e^3$

$y_4 = y_3 + e^3 = (y_1 + e^1 + e^2) + e^3$

$$\boxed{y_n = y_1 + \sum_{k=1}^{n-1} e^k}$$

$q = e > 1$

$S_n = e^1 + e^2 + \dots e^{n-1}$

$e S_n = e^2 + e^3 + \dots + e^n$

$(e-1) S_n = e^n - e^1$

$S_n = \dfrac{e^n - e}{(e-1)}$

$$y_n = 1 + \frac{e^n - e}{e-1}$$

$$y_n = \frac{\cancel{e} - 1 + e^n - \cancel{e}}{e-1} =$$

$$\boxed{y_n = \frac{e^n - 1}{e-1}} \quad \forall\ n \geq 1.$$

$\Delta : \mathcal{F} \longrightarrow \mathcal{F}$

$\{a_n\}_{n \in \mathbb{N}} \longrightarrow \{\Delta a_n\}_{n \in \mathbb{N}}$

$a_n \longrightarrow \Delta a_n = a_{n+1} - a_n$

Extra:

$\boxed{\Delta b_k = b_{k+1} - b_k}$

$\displaystyle\sum_{k=1}^{n} \Delta b_k = \Delta b_1 + \Delta b_2 + \dots + \Delta b_{n-1} + \Delta b_n$

$= (\cancel{b_2} - b_1) + (\cancel{b_3} - \cancel{b_2}) + \cancel{\dots} + (\cancel{b_n} - \cancel{b_{n-1}}) + (b_{n+1} - \cancel{b_n})$

$= b_{n+1} - b_1$

1707b) $\quad y_{n+1} - y_n = \sin(n) \qquad n \geq 0 \quad y_0 = 0$

$$\begin{cases} y_{n+1} = 1 y_n + \sin(n) \\ y_0 = 0 \end{cases}$$

$y_0 = 0$

$y_1 = y_0 + \sin(0)$

$y_2 = y_1 + \sin(1) = (y_0 + \sin(0)) + \sin(1) = y_0 + \sin(0) + \sin(1)$

$$y_2 = y_1 + \sin(1) = (y_0 + \sin(0)) + \sin(1) = y_0 + \sin(0) + \sin(1)$$

$$y_3 = y_2 + \sin(2) \qquad y_3 = y_0 + \sin(0) + \sin(1) + \sin(2)$$

$$y_3 = y_0 + \sin(0) + \sin(1) + \sin(2)$$

$$\boxed{y_n = 1 y_0 + \sum_{k=0}^{n-1} \sin(k)} \quad \circledast$$

$$\circledast \quad \sum_{k=0}^{n-1} \sin(k) = \sin(0) + \sin(1) + \sin(2) + \dots + \sin(n-1)$$

Hint:

$$\cos(A+B) = \cos A \cos B - \sin(A)\sin B$$

$$\cos(A-B) = \cos(A)\cos(-B) - \sin(A)\sin(-B) =$$

$$= \cos(A)\cos B + \sin(A)\sin(B)$$

$$\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$$

$$A = k, \quad B = \tfrac{1}{2} \Rightarrow \underbrace{\cos(k-\tfrac{1}{2}) - \cos(k+\tfrac{1}{2})}_{\Delta b_k} = 2\sin(k)\cdot\sin(\tfrac{1}{2})$$

$$\sum_{k=0}^{n-1} 2\sin(\tfrac{1}{2})\sin k = 2\sin(\tfrac{1}{2}) \sum_{k=0}^{n-1} \sin k = \sum_{k=0}^{n-1} \cos(k-\tfrac{1}{2}) - \cos(k+\tfrac{1}{2})$$

$$\sum_{k=0}^{n-1} \sin(k) = \frac{1}{2\cdot\sin(\frac{1}{2})} \underbrace{\sum_{k=0}^{n-1} \cos(k-\tfrac{1}{2}) - \cos(k+\tfrac{1}{2})}_{\circledast\circledast}$$

$\circledast\circledast$ Lösning:

$$\sum_{k=0}^{n-1} (\ ) - (\ ) = \cos(-\tfrac{1}{2}) - \cancel{\cos(0+\tfrac{1}{2})} +$$

$$\cancel{\cos(1-\tfrac{1}{2})} - \cancel{\cos(1+\tfrac{1}{2})} +$$

$$\cancel{\cos(2-\tfrac{1}{2})} - \cancel{\cos(2+\tfrac{1}{2})} +$$

$$\cancel{\cos(3-\tfrac{1}{2})} - \cancel{\cos(3+\tfrac{1}{2})} + \dots$$

$$\cancel{\cos((n-1)-\tfrac{1}{2})} - \cos((n-1)+\tfrac{1}{2}) =$$

$$= \cos(-1/2) - \cos((n-1)+1/2)$$

$$= \cos(1/2) - \cos(n-1/2)$$

(1°) $$\sum_{k=1}^{n-1} \sin(k) = \frac{1}{2\sin(1/2)}\left[\underbrace{\cos(1/2) - \cos(n-1/2)}_{*}\right]$$

$$\cos\left(-\tfrac{1}{2}\right) = \cos\left(\overset{A}{-\tfrac{n}{2}} \overset{B}{+\tfrac{n}{2} - \tfrac{1}{2}}\right) = \cos\left(-\tfrac{n}{2}\right)\cos\left(\tfrac{n}{2}-\tfrac{1}{2}\right) + \sin\left(\tfrac{n}{2}\right)\sin\left(\tfrac{n}{2}-\tfrac{1}{2}\right)$$

$$\cos\left(n-\tfrac{1}{2}\right) = \cos\left(\tfrac{n}{2}+\tfrac{n}{2}-\tfrac{1}{2}\right) = \cos\left(\tfrac{n}{2}\right)\cos\left(\tfrac{n}{2}-\tfrac{1}{2}\right) - \sin\left(\tfrac{n}{2}\right)\sin\left(\tfrac{n}{2}-\tfrac{1}{2}\right)$$

$$\cos(1/2) - \cos(n-1/2) = 2\sin\left(\tfrac{n}{2}\right)\sin\left(\tfrac{n}{2}-\tfrac{1}{2}\right)$$

$$\sum_{k=0}^{n-1} \sin(k) = \frac{\cancel{2}\sin\left(\tfrac{n}{2}\right)\sin\left(\tfrac{n}{2}-\tfrac{1}{2}\right)}{\cancel{2}\sin\left(\tfrac{1}{2}\right)}$$

1707 c) $y_{n+1} - 2y_n = n \quad n \geq 0 \quad y_0 = 0$

$$\boxed{y_{n+1} = 2y_n + n}$$

$$y_0 = 0$$

$y_0 = 0$

$y_1 = \underbrace{2y_0 + 0}$

$y_2 = 2(y_1) + 1 = 2(2y_0 + 0) + 1$

$= 2^2 y_0 + 2.0 + 1$

$y_3 = 2(y_2) + 2 = 2(2^2 y_0 + 2.0 + 1.) + 2$

$= \underbrace{2^3 y_0 + 2^2.0 + 2^1.1 + 2}$

$$y_4 = 2(y_3) + 3 = 2^4 y_0 + \underbrace{2^3 \cdot 0 + 2^2 \cdot 1 + 2^1 \cdot 2 + 2^0 \cdot 3}$$

$$\downarrow$$

$$y_n = 2^n y_0 + \underbrace{\boxed{\sum_{k=0}^{n-1} 2^k \cdot k}}_{(*)}$$

Nu behöver vi att hitta $(*)$

$$S = \sum_{k=1}^{n} 2^k \cdot k = 2^1 \cdot 1 + 2^2 \cdot 2 + 2^3 \cdot 3 + 2^4 \cdot 4 + \ldots + 2^n \cdot n$$

$$2S = 2\sum_{k=1}^{n} 2^k k = 2^2 \cdot 1 + 2^3 \cdot 2 + 2^4 \cdot 3 + 2^5 \cdot 4 + \ldots + 2^{n+1} \cdot n$$

$$(2S - S) = 2^{n+1} \cdot n + \ldots + 2^4(3-4) + 2^3(2-3) + 2^2(1-2) - 2^1 \cdot 1$$

$2^n(-1)$

$(n-2-(n-1))$

$(n-2-n+1)$

$$= 2^{n+1} \cdot n - 2^n - 2^{n-1} - \ldots - 2^2 - 2^1$$

$$= 2^{n+1} \cdot n - \left(2^n + 2^{n-1} \ldots + 2^2 + 2^1\right)$$

$$= n 2^{n+1} - \left[\frac{2^{n+1} - 2}{2-1}\right]$$

$$= n 2^{n+1} - 2^{n+1} + 2$$

$$S_n = (n-1) 2^{n+1} + 2$$

$$\boxed{S_{n-1} = (n-2) 2^n + 2}$$

$$\boxed{y_n = 2^n y_0 + 2 + (n-2) 2^n}$$

$$\underbrace{\sum_{k=1}^{n} 1}_{} = \underbrace{1 + 1 + 1 \cdots + 1}_{n} = n \cdot 1$$

$a_n = n$ $$\underbrace{\sum_{k=1}^{n} k^1}_{} = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$a_n = n^2 \rightarrow$ $$\boxed{\sum_{k=1}^{n} k^2 = ?} \qquad \underbrace{\alpha n^3 + \beta n^2 + \gamma n + \sigma}_{}$$

$$\sum_{k=1}^{n} k^3 = \alpha n^4 + \cdots$$

$$\sum_{k=1}^{n} k^m = \beta_{m+1} n^{m+1} + \cdots$$

$$\sum_{k=1}^{n} k^2 = ?$$

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$$\underbrace{\sum_{k=1}^{n} (k+1)^3}_{} = \sum_{k=1}^{n} (k)^3 + 3\underbrace{\sum_{k=1}^{n} k^2}_{?} + 3\underbrace{\sum_{k=1}^{n} k}_{\shortparallel} + \underbrace{\sum_{k=1}^{n} 1}_{}$$

$$\cancel{2^3} + \cancel{3^3} + \cdots + \cancel{+}(n+1)^3 = 1^3 + \cancel{2^3} + \cdots \cancel{n^3} + 3y + \underbrace{3 \cdot \frac{n(n+1)}{2} + n}_{}$$

$$\frac{1}{3}\left[(n+1)^3 - 1^3 - \frac{3}{2}(n)(n+1) - n\right] = y$$

$$\sum_{k=1}^{n} k^2 = y = \frac{1}{3}\left[(n+1)^3 - (n+1) - \frac{3}{2} n(n+1)\right]$$

$$y = \frac{1}{3}(n+1)\left[(n+1)^2 - 1 - \frac{3}{2} n\right]$$

$$y = \frac{1}{3}(n+1)\left[\frac{2n^2 - 4n - 3n}{2}\right]$$

$$y = \frac{1}{3}(n+1)\left[\frac{2n^2+n}{2}\right] =$$

$$y = \frac{1}{6}(n+1)\,n\,(2n+1)$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{y_{n+1} - \boxed{1}\,y_n = \underbrace{n^2}_{f(n)}}$$

Observera att

$\boxed{1} \to \boxed{2} \to g(n)$

Det kunde vara en andra funktion också.

Då kanske behöver vi ett starkare resultat:

Det finns en sats för den där typ av differensekvation

Tack för idag!