Ö8_V4_22april

den 22 april 2021 14:59

ELW 1709a / 1709b / 1711 / 1715c / 1717b

Sidan 47

1709 a) $y_{n+2} + y_{n+1} - 6y_n = 0 \quad (*)$

Lösning: Vi vill hitta $\{y_n\}_{n=1}^{\infty}$ så att y_n satsfiera ekvation $(*)$

Ansats $y_n = r^n$, $r \neq 0$

$ax^2 + bx + c = 0$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4.ac}}{2.a}$

Karaktenstika ekv. $\underbrace{r^2 + 1r^1 - 6}\, = 0$

$$r_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4.1.(-6)}}{2.1}$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} \quad \begin{cases} \dfrac{-1+5}{2} = \dfrac{4}{2} = 2 = r_1 \\ \dfrac{-1-5}{2} = \dfrac{-6}{2} = -3 = r_2 \end{cases}$$

$$\boxed{y_n = A\,2^n + B\,(-3)^n}$$

$\left\{ \begin{array}{l} \text{Lösning } y_n \in \underline{\text{Span}\{2^n, (-3)^n\}} = U \\ \dim U = 2 \end{array} \right.$

$\square$

1709 c) $\underbrace{y_{n+2} - 4y_{n+1} + 4y_n} = 0$

Lösning: Ansatts $y_n = r^n$

$$r^2 - 4r + 4 = 0$$

$$r_{1,2} = \frac{+4 \pm \sqrt{16 - 4.1.4}}{2} = \frac{4 \pm 0}{2} = 2 \qquad \underline{\underline{r_1 = r_2}}$$

$$\cancel{y_n =} \underbrace{A\,2^n + B\,2^n}_{\text{linjärt Beroende}} = (A+B)\,2^n = C\,2^n$$

$$y_n = A\,2^n + B\,n\,2^n \qquad U = \operatorname{span}\{2^n, n2^n\}$$

Obs. $\boxed{y_{n+3} + \alpha y_{n+2} + \beta y_{n+1} + \gamma y_n = 0}$

$$r^3 + \alpha r^2 + \beta r + \gamma = 0 \longrightarrow r_1 = r_2 = r_3 = 2.$$

$$y_n = \cancel{\;} A(2)^n + B(2)^n + C(2)^n$$

$$\boxed{y_n = A(2)^n + Bn(2)^n + Cn^2 2^n}$$

1711) $\{y_n\}_{n=1}^{\infty}$ $\boxed{y_{n+2} = \frac{1}{2} y_{n+1} + \frac{1}{2} y_n}$

1) Hitta y_n lösning

2) $\lim_{n\to\infty} y_n$

1) $y_{n+2} - \frac{1}{2} y_{n+1} - \frac{1}{2} y_n = 0$

Karakt. Ekv: $r^2 - \frac{1}{2} r - \frac{1}{2} = 0 \qquad r_1 = 1 \qquad r_2 = \left(-\frac{1}{2}\right)$

$$\boxed{y_n = A(1)^n + B\left(-\frac{1}{2}\right)^n}$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} A(1)^n + \cancel{B\left(-\frac{1}{2}\right)^n} \to 0 = A(1)^n = A$$

Att bestäma A, B behöver vi ha:

$$\begin{cases} y(0) = y_0 \\ y(1) = y_1 \end{cases}$$

$$y_n = \cancel{C_1}\,A\,(1)^n + \cancel{C_2}\,B\left(-\tfrac{1}{2}\right)^n$$

$y(1) = y_1$

$y(2) = y_2$

$$\begin{cases} y(1) = C_1(1)^1 + C_2\left(-\tfrac{1}{2}\right) = y_1 \\ y(2) = C_1(1)^2 + C_2\left(-\tfrac{1}{2}\right)^2 = y_2 \end{cases}$$

$$\begin{cases} 1\,C_1 - \tfrac{1}{2}C_2 = y_1 \\ 1\,C_1 + \tfrac{1}{4}C_2 = y_2 \end{cases} \Rightarrow C_1 = y_1 + \tfrac{1}{2}C_2$$

substituera.

$$\underbrace{\left(y_1 + \tfrac{1}{2}C_2\right) + \tfrac{1}{4}C_2}\ = y_2$$

$$\left(\tfrac{2}{2}\tfrac{1}{2} + \tfrac{1}{4}\right)C_2 = y_2 - y_1$$

$$\left(\tfrac{3}{4}\right)C_2 = (y_2 - y_1)$$

$$\boxed{C_2 = \tfrac{4}{3}(y_2 - y_1)}$$

$$\cancel{C}B_2 = -\tfrac{4}{3}y_1 + \tfrac{4}{3}y_2$$

$$C_1 = y_1 + \tfrac{1}{\cancel{2}}\cdot\tfrac{\cancel{4}}{3}(y_2 - y_1)$$

$$C_1 = y_1 + \underbrace{\tfrac{2}{3}(y_2 - y_1)}$$

$$C_1 = \left(1 - \tfrac{2}{3}\right)y_1 + \tfrac{2}{3}y_2$$

$$\boxed{\cancel{C_1}\,A = \tfrac{1}{3}y_1 + \tfrac{2}{3}y_2}$$

$$\lim_{n\to\infty} y_n = A = \underline{\tfrac{1}{3}y_1 + \tfrac{2}{3}y_2} = \tfrac{1}{3}(2y_2 + y_1) \quad \square$$

1715c $\left[\; y_{n+2} - 3y_{n+1} - 10y_n = \underbrace{36n - 21}\right.$

Lösning: $y_n = y_n^H + y_n^P$

i) y_n^H lösning till Homogena Differensekv

ii) y_n^P partikulär lösning.

i) $y_{n+2} - 3y_{n+1} - 10y_n = 0$

$$\downarrow$$

$r^2 - 3r - 10 = 0$

$$r_{1,2} = \frac{3 \pm \sqrt{9+4.1.(+10)}}{2} = \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2}$$

$$r_1 = \frac{3+7}{2} = \frac{10}{2} = 5 \qquad r_2 = \frac{3-7}{2} = -\frac{4}{2} = -2$$

$$y_n^H = \alpha 5^n + \beta (-2)^n$$

ii) $1 y_{n+2} - 3y_{n+1} - 10y_n = \underbrace{36n - 21}_{\text{polynom}}$

$\{$polynom, 5^n, $(-2)^n\}$ linjärt Oberoende.

Då Ansats $\boxed{y_n^P = An + B}$

$$1\ y_{n+2} = A(n+2) + B = 1An + (2A + B)$$

$$-3\ y_{n+1} = -3(A(n+1) + B) = -3An - 3(A + B)$$

$$-10\ y_n = -10(An + B) = -10An + (-10B)$$

$$36n - 21 = (-12A)n + (+2A - 3A + B - 3B - 10B)$$

$$(-A - 12B)$$

$$\begin{cases} -12A = 36 & \Rightarrow \quad A = -36/12 = -3 \\ -A - 12B = -21 \end{cases}$$

$$+(+3) - 12B = -21$$

$$-12B = -21 - 3 = -24$$

$$-12B = -24 \Rightarrow \boxed{B = 2}$$

$$\boxed{y_n^P = -3n + 2.}$$

$$\boxed{y_n^P = -3n + 2}$$

$$y_n = y_n^H + y_n^P$$

$$\boxed{y_n = A\,5^n + B(-2)^n - 3n + 2}$$

1717b

$$\begin{cases} y_{n+2} - 2y_{n+1} + 4y_n = \underbrace{n\,2^n}_{(i)} + \underbrace{4^n}_{(ii)} \\ y(0) = 1/3 \qquad y(1) = 7/3 \end{cases}$$

Lösning:

1°) $y_n^H = A\,r_1^n + B\,r_2^n$

2°) $y_n^{P_1}$ för $\longrightarrow n\,2^n$

3°) $y_n^{P_2}$ för $\longrightarrow 4^n$

4°) $y_n = y_n^H + y_n^{P_1} + y_n^{P_2}$

5°) $\begin{cases} y(0) = 1/3 \\ y(1) = 7/3 \end{cases}$ lösa system att bestämma A, B

i) $y_{n+2} - 2y_{n+1} + 4y_n = 0$

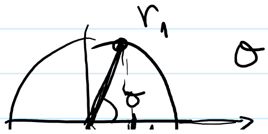
$$r^2 - 2r + 4 = 0 \qquad r_{1,2} = \frac{+2 \pm \sqrt{4 - 4.1.4}}{2.1} =$$

$$r_{1,2} = \frac{2 \pm \sqrt{4-16}}{2.1} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$\boxed{r_1 = 1 + \sqrt{3}\,i} \qquad \boxed{r_2 = 1 - \sqrt{3}\,i} \qquad \boxed{r_2 = \overline{r_1}} \qquad \|r_1\| = \|r_2\|$$

$$\|r_1\| = \sqrt{1+3} = \sqrt{4} = 2$$

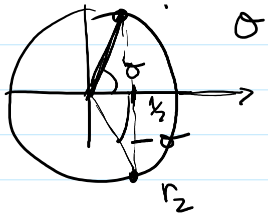
$$r_1 = \frac{2}{2}\left(1 + \sqrt{3}\,i\right) = \frac{2}{1}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)$$



$$\theta = \frac{\pi}{3} = 60^{\circ}$$

$r_1 = \frac{2}{2}\left(1+\sqrt{3}\,i\right) = 2\left(\frac{1}{2} + \underbrace{\frac{\sqrt{3}}{2}}\, i\right)$

$\rho\left(\cos\theta + i\sin\theta\right)$



$\theta = \frac{\pi}{3} = 60^\circ$

$r_1 = 2\left(\cos \pi/3 + i\sin \pi/3\right) = \underbrace{2}\, e^{i\pi/3}$

$r_2 = 2\left(\cos \pi/3 - i\sin \pi/3\right) = \underbrace{2}\, e^{-i\pi/3}$

$y_n^H = A(r_1)^n + B(r_2)^n$

$y_n^H = A2^n e^{in\pi/3} + B\cdot 2^n e^{-in\pi/3}$

$= A2^n\left(\cos n\frac{\pi}{3} + i\sin n\frac{\pi}{3}\right) + B2^n\left(\cos n\frac{\pi}{3} - i\sin n\frac{\pi}{3}\right)$

$\underbrace{(A+B)}2^n\cos n\frac{\pi}{3} + (iA - iB)2^n\sin n\frac{\pi}{3}$

$$\boxed{y_n^H = C\,2^n\cos n\pi/3 + D\,2^n\sin n\pi/3}$$

ii) $y_{n+2} - 2y_{n+1} + 4y_n = n\cdot 2^n = (1n+0)2^n$

$\{\text{polynom}\cdot 2^n,\ \cos n\pi/3\cdot 2^n,\ \sin n\pi/3\cdot 2^n\}$ linjärt Oberoende

$$\boxed{y_n^{P_1} = (An+B)2^n}$$

$y_{n+2} = \left(A(n+2)+B\right)2^{n+2} = (An+2A+B)\,2^n\, 2^2 =$ ($2^2 = 4$)

$y_{n+2} = (4An+8A+4B)\,2^n$

$y_{n+1} = \left(A(n+1)+B\right)2^{n+1} = \left(An + (A+B)\right)2^n\cdot 2^1$

$y_{n+1} = \left(2An + (2A+2B)\right)2^n$

$$y_n = (An + B)2^n$$

$$\begin{aligned}
y_{n+2} &= (4A n + (8A + 4B))2^n \\
-2y_{n+1} &= -2(2An + (2A+2B))2^n \\
+4y_n &= 4(An + B)2^n
\end{aligned}$$

$$(1n+0)2^n = \left(n(\cancel{4A} - \cancel{4A} + 4A) + (8A + \cancel{4B} - 4A - \cancel{4B} + 4B)\right)2^n$$

$$(1n+0)\cancel{2^n} = ((4A)n + (4A+4B))\cancel{2^n}$$

$$\begin{cases} 4A = 1 \Longrightarrow A = 1/4 \\ \cancel{4}A + \cancel{4}B = 0 \Rightarrow B = -A \Longrightarrow B = -1/4 \end{cases}$$

$$y_n^{P_1} = \left(\frac{1}{4}n - \frac{1}{4}\right) \cdot 2^n$$

3°) $y_{n+2} - 2y_{n+1} + 4y_n = 1 \, 4^n$ konstant = polynom av grad <u>noll</u>

Ansatt $y_n^{P_2} = A \cdot 4^n$

$$\begin{aligned}
y_{n+2} &= A\,4^{n+2} = A \cdot 4^2 \cdot 4^n = 16A\,4^n \\
y_{n+1} &= A\,4^{n+1} = A \cdot 4 \cdot 4^n = 4A\,4^n \\
y_n &= A\,4^n = 1A\,4^n
\end{aligned}$$

$$\begin{aligned}
y_{n+2} &\quad 16A\,4^n \\
-2y_{n+1} &\quad -2(4A)\,4^n
\end{aligned}$$

$$\begin{array}{lcl} -2y_{n+1} & & -2(4A)\,4^n \\ +4y_n & = & +4(1\cdot A)\,4^n \\ \hline 1\cdot 4^n & & (\underbrace{16A-8A+4A}) 4^n \end{array}$$

$12A = 1$ $\quad \boxed{A = \frac{1}{12}}$

$$y_n^{P_2} = \frac{1}{12}4^n$$

$$y_n = y_n^H + y_n^{P_1} + y_n^{P_2}$$

$$\boxed{y_n = \alpha 2^n \cos n\pi/3 + \beta 2^n \sin n\pi/3 + \left(\frac{1}{4}n - \frac{1}{4}\right)2^n + \frac{1}{12}4^n}$$

4) Lösa system att bestämma α, β.

$$\begin{cases} y_0 = 1/3 \\ y_1 = 7/3 \end{cases}$$

$$\begin{cases} y_0 = \alpha 2^0 \cdot 1 + \cancel{\beta \cdot 2^0\, 0} + \left(0 - \frac{1}{4}\right)\cancel{2^0} + \frac{1}{12}\cancel{4^0} = 1/3 \\ y_1 = \alpha \cancel{2^1} \frac{1}{\cancel{2}} + \beta \cdot \cancel{2^1} \frac{\sqrt{3}}{\cancel{2}} + \cancel{\left(\frac{1}{4} - \frac{1}{4}\right)2^1} + \frac{1}{\cancel{12}}\cancel{4^1} = 7/3 \end{cases}$$

$$\begin{cases} \alpha - 1/4 + \frac{1}{12} = 1/3 & \Rightarrow \alpha = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2} \\ \alpha + \sqrt{3}\beta + \frac{1}{3} = 7/3 \end{cases}$$

$$\sqrt{3}\beta = \frac{7}{3} - \frac{1}{3} - \alpha = \frac{6}{3} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\beta = \frac{1}{\cancel{\sqrt{3}}}\left(\frac{\cancel{3}}{2}\right)\frac{\sqrt{3}}{\cancel{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

$$y_n = 2^n\left(\frac{1}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{3}\sin\frac{n\pi}{3}\right) + \underbrace{\left(\frac{n}{4}-\frac{1}{4}\right)2^n}_{} + \underbrace{\frac{1}{12}4^n}_{}$$

$$\frac{1}{2^2}(n-1)2^n + \frac{1}{4.3}4^n$$

$$\underbrace{(n-1)2^{n-2}}_{} + \underbrace{\frac{1}{3}4^{n-1}}_{}$$

1717e) $y_{n+2} - 2y_{n+1} + 4y_n = \underbrace{2^n \sin n\pi/3}_{}$ (Nästa vecka!)

Uppgift: 1706 c) sidan 41.

$$\begin{cases} 2y_{n+1} - y_n = 4 \\ y_0 = 3 \end{cases}$$

Lösning:

$$\begin{cases} y_{n+1} = \frac{1}{2}y_n + 2 \\ y_0 = 3 \end{cases} \quad \forall n \in \mathbb{N}.$$

$$y_0 = 3$$

$$y_1 = \underbrace{\frac{1}{2}y_0 + 2}_{}$$

$$y_2 = \frac{1}{2}(y_1) + 2 = \frac{1}{2}\left(\frac{1}{2}y_0 + 2\right) + 2 =$$

$$= \frac{1}{2^2}y_0 + \frac{1}{2}\cdot 2 + 2$$

$$y_3 = \frac{1}{2}y_2 + 2 = \frac{1}{2}\left(\frac{1}{2^2}y_0 + \frac{1}{2}\cdot 2 + 2\right) + 2$$

$$= \underbrace{\frac{1}{2^3}y_0}_{} + \underbrace{\frac{1}{2^2}\cdot 2 + \frac{1}{2^1}2 + \frac{1}{2^0}2}_{}$$

$$y_4 = \frac{1}{2^4}y_0 + \underbrace{\frac{1}{2^3}2 + \frac{1}{2^2}\cdot 2 + \frac{1}{2^1}2 + \frac{1}{2^0}2}_{}$$

$$y_4 = \frac{1}{2}(y_3) + 2$$

$$= \frac{1}{2^4}y_0 + \frac{1}{2^3}2 + \frac{1}{2^2}2 + \frac{1}{2^1}2 + \frac{1}{2^0}2$$

$$\boxed{y_n = \frac{1}{2^n}y_0 + 2\underbrace{\sum_{k=0}^{n-1}\frac{1}{2^k}}_{}}$$

$$\frac{}{2^4} y_0 \quad \frac{}{2^3} \quad \frac{}{2^2} \quad \frac{}{2^1} \quad \frac{}{2^0}$$

$$Y_n = \frac{1}{2^n} y_0 + 2 \underbrace{\sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k}$$

$$S = \sum_{k=0}^{n-1} q^k = q^0 + q^1 + \dots + q^{n-1}$$

$$qS = q^1 + q^2 + \dots + q^n$$

$$(1-q)S = 1 - q^n$$

$$S = \frac{(1-q^n)}{(1-q)}$$

$$Y_n = \frac{1}{2^n} \cdot y_0 + 2 \left[\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \right]$$

$$1 - \frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{\frac{1}{2}} = 2$$

$$Y_n = \frac{1}{2^n} y_0 + 2 \cdot 2 \left[1 - \left(\frac{1}{2}\right)^n \right]$$

$$Y_n = \left(\frac{1}{2}\right)^n (y_0 - 4) + 4$$

$$y_0 = 3$$

$$Y_n = \left(\frac{1}{2}\right)^n (3 - 4) + 4$$

$$\boxed{Y_n = -1 \left(\frac{1}{2}\right)^n + 4}$$