L10_V6_03maj2021

den 4 maj 2021 13:41

Kapitel 19 Potensserier.

$$\underbrace{\sum_{k=0}^{\infty} a_k (x-a)^k}$$ $\{a_k\}$ talföljd

$x, a \in I \subseteq \mathbb{R}$

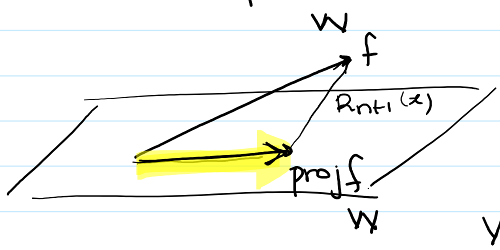
§ 19.1 Taylorserier ($a=0$ Maclaurinserier)

Antag f är funktion som har derivator av alla ordningar i $I \subseteq \mathbb{R}$

$$f \in \underbrace{C^{\infty}(I)}, \quad I \subseteq \mathbb{R} \qquad C^{\infty}(I) = \text{Vektorrum}$$

Om $a \in I$, $x \in I$

$$\underbrace{f(x)} = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \underbrace{(x-a)^k}}_{\text{proj } f(x)} + \underbrace{R_{n+1}(x)}_{\text{Lagrange Rest}}$$



$W = \text{Span}\{(x-a)^0, (x-a)^1, (x-a)^2, \dots (x-a)^n\}$

$$R_{n+1}(x) = f(x) - P_n(x)$$

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad \xi \in (x, a) \subseteq I$$

(Adams Bok En Variabel Analys (Calculus))

ELW : θx, $0 < \theta < 1$ $\quad \theta x \in I$.

Theorem Adams Bok sidan 280.

If $f(x) = Q_n(x) + O((x-a)^{n+1})$ as $x \to a$

$Q_n(x)$ = polynomial of degree $\leq n$.

Then $Q_n(x) = P_n(x)$ = Taylor polynom. for $f(x)$ at $x=a$.

Then $Q_n(x) = P_n(x)$ = Taylor polynom. for $f(x)$ at $x=a$.

Exampel. $$P_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$ Taylor polinom

när $a=0$, då kallas P_n ~~som~~ Maclaurinpolynom.

$$f(x) = \underbrace{f(a) + \frac{f'(a)}{1!}(x-a)^1 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}}_{\text{Error } R_{n+1}(x)}$$

$\to 0$

$f(x) = e^x$, Maclaurinserier

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2})$$

$$e^{(-x)} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots + \frac{(-x)^{2n+1}}{(2n+1)!} + O((-x)^{2n+2})$$

$$e^{(-x)} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + -\frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2})$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\begin{array}{l} 1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \dots \quad O(x^{2n+2}) \\ 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \dots \quad O(x^{2n+2}) \end{array}\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2} + \cancel{2}\cdot\frac{x^2}{2!} + \cancel{2}\frac{x^4}{4!} + \cancel{2}\frac{x^6}{6!} \dots + O(x^{2n+2})\right)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + O(x^{2n+2})$$

Exempel: $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \dots (-1)^n\frac{x^n}{n} + O(x^{n+1})$$

Exempel: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \ldots (-1)^n \frac{x^n}{n} + O(x^{n+1})$

Bestäm Taylor Polynom 3:e grad för $\ln(x)$ kring $x = e$

$a = e$.

Lösning: $\begin{cases} x = e + (x-e) = e\left(1 + \frac{(x-e)}{e}\right) = e(1+t) \\ t = \frac{x-e}{e} \qquad x \to e \Rightarrow t \to 0 \end{cases}$

$$\ln(x) \overset{*}{=} \ln(e(1+t)) = \ln e + \underbrace{\ln(1+t)}_{\text{Taylor Polynom}} =$$

$$= \ln(e) + \left[t - \frac{t^2}{2} + \frac{t^3}{3}\right] + O(t^4)$$

$$= 1 + \left[\frac{(x-e)}{e} - \frac{1}{2}\left(\frac{x-e}{e}\right)^2 + \frac{1}{3}\left(\frac{x-e}{e}\right)^3\right] + O\left(\left(\frac{x-e}{e}\right)^4\right)$$

$$P_3(x) = 1 + \left(\frac{x-e}{e}\right) - \frac{1}{2}\left(\frac{x-e}{e}\right)^2 + \frac{1}{3}\left(\frac{x-e}{e}\right)^3$$

$$\ln(x) = P_3(x) + O((x-e)^4)$$

Sidan 85 Binomialserier.

$$(1+x)^p \overset{*}{=} \sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots + \ldots, \text{ då } |x| < 1.$$

(?)

för all $p \in \mathbb{R}$

Obs: När $p \in \mathbb{N}$.

$$(1+x)^p = \sum_{k=0}^{p} \binom{p}{k} x^k = \binom{p}{0} 1^p \cdot x^0 + \binom{p}{1} 1^{p-1} \cdot x^1 + \binom{p}{2} x^2 + \ldots + \binom{p}{p} x^p$$

$\binom{p}{\ } = \quad p!$

$$\sum_{k=0}$$

$$\binom{p}{k} = \frac{p!}{(p-k)!\,k!}$$

<u>Obs</u>: $p \in \mathbb{R} \setminus \mathbb{N}$

$$\binom{p}{k} = \frac{\overset{\in\mathbb{R}}{p}\cdot\overset{\in\mathbb{R}}{(p-1)}\overset{\in\mathbb{R}}{(p-2)}\cdots\overset{\in\mathbb{R}}{(p-(k-1))}}{k!}$$

$p = \pi \qquad k = 0, 1, \dots \quad n \in \mathbb{N}$

<u>Sats</u>: 19:3 Hjälpsats om Potensserier Konvergens.

Om $\sum_{k=0}^{\infty} a_k x^k$ konvergerar i punkt $x_0 \neq 0$

$\left(\sum_{k=0}^{\infty} a_k x_0^k = \sum_{k=0}^{\infty} b_k \right.$

Så konvergerar $\sum_{k=0}^{\infty} a_k x^k$ för alla x sådant $|x| < |x_0|$.