Ö11_V6_5maj2021

den 5 maj 2021 12:53

Ö10 : 1902 f

Ö11 : 1903 a / 1904 b / 1908 a / 1914 a

1902) Bestäm konvergensintervallen för

f) $\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k$ $\qquad a_k = \frac{k^k}{k!}$

Tanka

1) Rotformeln
$\sqrt[k]{|a_k|} \xrightarrow[k\to\infty]{} H \Rightarrow R = 1/H$

2) Kvotformeln
$\left|\frac{a_{k+1}}{a_k}\right| \to K \Rightarrow R = \frac{1}{K}$

$I = (-R, R)$
$|x| < R$

? $x = R$, $x = -R$?

Enligt Kvotformel, börjar vi att räkna:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} =$$

$$= \frac{(k+1)^k \cdot \cancel{(k+1)}}{\cancel{(k+1)} \cdot \cancel{k!}} \cdot \frac{\cancel{k!}}{k^k} =$$

$$= \frac{(k+1)^k}{k^k} = \left(\frac{k+1}{k}\right)^k =$$

$$= \left(1 + \frac{1}{k}\right)^k$$

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k \underset{\otimes}{=} e = K \Rightarrow R = \frac{1}{K} = \frac{1}{e}$$

Då har vi $I = \left(-\frac{1}{e}, \frac{1}{e}\right)$ som konvergensintervallen, enligt Kvotformeln Satsen 19.5.

Men vi behöver också analysera vad händer när $x = -\frac{1}{e}$, $x = \frac{1}{e}$.

ii) $x = \frac{1}{e} \Rightarrow \sum_{k=1}^{\infty} a_k \left(\frac{1}{e}\right)^k = \sum_{k=} \underbrace{\left(\frac{k^k}{k!}\right) \frac{1}{e^k}}_{b_k}$

$$b_k = \left(\frac{k}{e}\right)^k \underset{\circledast}{\frac{1}{k!}}$$

$\otimes$ Stirlings formel

sats 18.3

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,(1+\varepsilon_n) \qquad \varepsilon_n \xrightarrow[n\to\infty]{} 0$$

Enligt stirlings formel har vi

$$b_k = \cancel{\left(\frac{k}{e}\right)^k} \frac{1}{\cancel{\left(\frac{k}{e}\right)^k} \sqrt{2\pi k}\,(1+\varepsilon_k)}$$

$$b_k = \frac{1}{\sqrt{2\pi k}\,(1+\varepsilon_k)} \qquad \varepsilon_k \xrightarrow[k\to\infty]{} 0$$

$$\lim_{k\to\infty} b_k = \frac{1}{\infty} = 0$$

Jämförelsekriterium

$\frac{b_k}{c_k} \xrightarrow[k\to\infty]{} \alpha \Rightarrow \Sigma b_k$ & Σc_k har samma beteende

$$c_k = \frac{1}{\sqrt{k}}$$

$$\lim_{k\to\infty} \frac{b_k}{c_k} = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{2\pi}\sqrt{k}} \frac{1}{(1+\varepsilon_k)}}{\frac{1}{\sqrt{k}}} = \frac{1}{\sqrt{2\pi}} = \alpha > 0$$

men $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} c_k$ är divergent

Då enligt Jämförelsekriterium är $\sum b_k$ också divergent.

Däremot $x = 1/e \notin I =$ konvergensintervall $[-\frac{1}{e}, \frac{1}{e})$? öppnet ?

iii) $X = -1/e$

$$\sum_{k=1}^{\infty} a_k \left(-\frac{1}{e}\right)^k = \sum_{k=1} (-1)^k \underbrace{a_k \frac{1}{e^k}}_{b_k}$$

alternerande serier

$$b_k = \frac{k^k}{k!} \frac{1}{e^k} = \left(\frac{k}{e}\right)^k \frac{1}{k!}$$ stirling formeln

$$b_k = \frac{1}{\sqrt{2\pi k}\,(1+\varepsilon_k)} \xrightarrow[k\to\infty]{} 0$$

$$\varepsilon_k \xrightarrow[k\to\infty]{} 0$$

Tanka

Leibniz kriterium

$\{b_k\}_{k=1}^{\infty}$ avtagande ?

$b_k \xrightarrow[k\to\infty]{} 0$

$\Downarrow$

$\sum (-1)^k b_k$ konvergerar

Fråga: Är $\{b_k\}$ avtagande ?

$b_k > b_{k+1} \quad \forall k \in \mathbb{N}$?

$b_k - b_{k+1} > 0 \quad \forall k \in \mathbb{N}$. ?

$b_k - b_{k+1} > 0 \Leftrightarrow \boxed{b_{k+1} - b_k < 0}$

Lösning: $b_{k+1} - b_k =$

$$\frac{(k+1)^{k+1}}{e^{k+1}} \frac{1}{(k+1)!} - \frac{k^k}{e^k} \frac{1}{k!} =$$

$$= \frac{(k+1)^k \cdot \cancel{(k+1)}}{e^{k+1}} \frac{1}{\cancel{(k+1)}\,k!} - \frac{k^k}{e^k} \cdot \frac{1}{k!} \cdot \frac{e}{e} =$$

$$\frac{(k+1)^k}{k!} \frac{1}{e^{k+1}} - \frac{k^k}{k!} \frac{e}{e^{k+1}} =$$

$$\frac{1}{k!\,e^{k+1}} \left[(k+1)^k - k^k e \right] =$$

$$\frac{k^k}{k!\,e^{k+1}} \left[\left(\frac{k+1}{k}\right)^k - e \right] =$$

$$\frac{k^k}{k!\,e^{k+1}} \left[\underbrace{\left(1 + \frac{1}{k}\right)^k}_{c_k} - e \right]$$

$c_k = \left(1 + \frac{1}{k}\right)^k$? Fråga är:

Hur går $c_k \to e$? växande eller avtagande ??

Vi behöver studera på vilket sätt $c_k \to e$.

$$c_{k+1} - c_k = \left(1 + \frac{1}{k+1}\right)^{k+1} - \left(1 + \frac{1}{k}\right)^k > ⊛$$

$$c_{k+1} - c_k = \left(1 + \frac{1}{k+1}\right)^{k+1} - \left(1 + \frac{1}{k}\right)^{k} > (*)$$

$$\left(1 + \frac{1}{k+1}\right)^{k+1} - \left(1 + \frac{1}{(k+1)}\right)^{k} =$$

$$\underbrace{\left(1 + \frac{1}{k+1}\right)^{k}\left(1 + \frac{1}{k+1}\right) - \left(1 + \frac{1}{k+1}\right)^{k}}$$

$$\left(1 + \frac{1}{k+1}\right)^{k}\left(\cancel{1} + \frac{1}{k+1} - \cancel{1}\right)$$

$$\left(1 + \frac{1}{k+1}\right)^{k}\left(\frac{1}{k+1}\right) > 0$$

$c_k \to e$

$0 < c_1 < c_2 < \dots < \dots e$ } växande

$c_k < e$

då är $\boxed{c_k - e < 0}$

då är $\{b_k\}_{k=1}^{\infty}$ avtangande och $\left(b_k \underset{k\to\infty}{\longrightarrow} 0\right)$

Enligt Leibnitzkriterium

$\sum (-1)^k b_k$ är konvergent.

Däremot $x = -\frac{1}{e} \in I \Rightarrow I = \left[-\frac{1}{e}, \frac{1}{e}\right)$

stängt öppnet. □

~~Ö11~~

1903. a) Vilken är I = konvergensintervall.

$$\sum_{k=0}^{\infty} (3^{-k})\, x^{2k} = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \cdot (x^2)^k$$

variabel byte $\boxed{t = x^2}$ $\boxed{x = \pm\sqrt{t}}$

$$\sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{3}\right)^k}_{a_k} t^k$$

Sats Rotformeln

$\sqrt[k]{|a_k|} \underset{k\to\infty}{\longrightarrow} H \Rightarrow R = \frac{1}{H}$

Lösning:

$a_k = \left(\frac{1}{3}\right)^k$

Och enligt Rotformeln ska vi studera

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\frac{1}{3}\right)^k} = \frac{1}{3} = H$$

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\tfrac{1}{3}\right)^k} = \frac{1}{3} = H$$

Då har vi $R = \dfrac{1}{\left(\frac{1}{3}\right)} = 3$

$$\boxed{|t| < 3}$$

$$\downarrow_2$$

$$x^2 < 3$$

$$\boxed{|x| < \sqrt{3}}$$

$$I = (-\sqrt{3}, \sqrt{3})$$

$$x \in I \qquad -\sqrt{3} < x < \sqrt{3}$$

Men vad händer när $x = -\sqrt{3}$, eller $x = \sqrt{3}$.

i) $x = \sqrt{3}$ $\quad t = 3$

$x = -\sqrt{3}$ $\quad t = 3$

$$\sum_{k=1}^{\infty} \left(\frac{1}{3^k}\right) \cdot (3)^k = \sum_{k=1}^{\infty} \underbrace{1}_{c_k} \qquad c_k \xrightarrow[k\to\infty]{} 1 \neq 0$$

$$c_k \not\to 0 \Rightarrow \sum c_k \text{ divergent.}$$

Då $x = \sqrt{3} \notin I$ –

$x = -\sqrt{3} \notin I$

$$I = (-\sqrt{3}, \sqrt{3})$$

$$-\sqrt{3} < x < \sqrt{3}$$

öppna.

$$\sum_{k=0}^{\infty} \frac{1}{3^k} x^{2k}$$ konvergent för $x \in (-3, 3)$

□

1904b) Bestäm konvergensradierna till följande potensserier.

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} x^n; \qquad a_n = \frac{(n!)^3}{(3n)!}$$

Sats 18.3 Stirlingsformeln.

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,(1+\varepsilon_n)$$

$$\varepsilon_n \xrightarrow[n\to\infty]{} 0$$

Lösning:

$$\sqrt[n]{|a_n|} = (a_n)^{1/n}$$

$a_n > 0$

Rotformeln

$$\sqrt[n]{|a_n|} \to H \Rightarrow R = \frac{1}{H}$$

$$(a_n)^{1/n} = \left[\frac{\left(\frac{n}{e}\right)^{3n} (2\pi n)^{3/2} (1+\varepsilon_n)^3}{\left(\frac{3n}{e}\right)^{3n} (2\pi\, 3n)^{1/2} (1+\varepsilon_{3n})}\right]^{1/n}$$

$$\varepsilon_n \to 0$$

$$\varepsilon_{3n} \to 0$$

$$n \to \infty$$

$$\left[\underbrace{\left(\frac{n}{e}\cdot\frac{e}{3n}\right)^{3n}}_{\left(\frac{1}{3}\right)^{3n}} \frac{(2\pi n)^{3/2 - 1/2}}{3^{1/2}} \cdot \frac{(1+\varepsilon_n)^3}{(1+\varepsilon_{3n})}\right]^{1/n}$$

$$\Big[\underbrace{\quad}_{\left(\frac{1}{3}\right)^{3n}} \quad \Big]$$

$$\left[\left(\frac{1}{3}\right)^{\cancel{3n}/\cancel{n}} \frac{(2\pi n)^{1/n}}{3^{1/2n}} \frac{(1+\varepsilon_n)^{3/n}}{(1+\varepsilon_{3n})^{1/n}}\right] =$$

$$\lim_{n\to\infty} \left[\frac{1}{3^{3+\frac{1}{2n}}} \underbrace{\sqrt[n]{2\pi}\,\sqrt[n]{n}}_{\text{sats 18.4}} \frac{(1+\varepsilon_n)^{3/n}}{(1+\varepsilon_{3n})^{1/n}}\right] =$$

$\frac{1}{2n} \to 0$, $\sqrt[n]{2\pi}\sqrt[n]{n} \to 1$, $(1+\varepsilon_n)^{3/n} \to 1$, $\varepsilon_n \to 0$, $(1+\varepsilon_{3n})^{1/n} \to 1$, $3n \to \infty$

Sats 18.4

1) $\lim_{n\to\infty} \sqrt[n]{a} = 1$

2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$= H = \frac{1}{3^3}$$

Då enligt Rot formel $\boxed{R = \frac{1}{H} = 3^3}$

Konvergensradien är $\underline{R = 27}$ □

1908 a) { Konvergensintervallet ? / Summan = ? (Hint Använd Termvis derivering)

a) $1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k} x^{2k}$

Lösning: Låt $x^2 = t \implies x^{2k} = t^k$

$$S(x) = 1 + \sum_{k=1}^{\infty} (-1)^k \underbrace{\frac{1}{2k}}_{a_k} t^k$$

sats kvotformeln

$$\left|\frac{a_{k+1}}{a_k}\right| \to K \implies R = 1/K$$

Enligt kvotformeln, analyserar vi:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{1}{(2k+1)}}{\frac{1}{2k}} = \frac{1}{\cancel{2}(k+1)}\frac{\cancel{2}k}{1} = \frac{1}{\frac{(k+1)}{k}} = \frac{1}{\left(1+\frac{1}{k}\right)}$$

$$\lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \qquad \boxed{R = 1}$$

$\frac{1}{k} \to 0$

$t^2 \to x = \pm\sqrt{t}$

$I = (-1, 1)$ kandidat

Vi fortfarande behöver analysera $x = -1$, $x = 1$ $(t^2 = 1)$

$x = 1, -1$ $(t^2 = 1)$

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \quad (*)$$

Leibniz kriterium

$\{b_k\}$ avtagande, $b_k \to 0$

$\Downarrow$

$\sum (-1)^k b_k$ är konv.

Då enligt Leibnizkriterium har vi att $(*)$ är konvergent

$$\left(\begin{array}{l} b_k = 1/k \longrightarrow 0 \quad \checkmark \\ \{b_k\} \text{ avtagande } \checkmark \end{array} \right)$$

$$R = 1 \qquad \underline{I = [-1, 1]}.$$

II. Summan. $\quad S(x) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k}$ konvergent. med $\underline{|x| \leq 1}$

<u>Derivation termvis</u>

$$\frac{dS(x)}{dx} = 0 + \sum_{k=1}^{\infty} (-1)^k \frac{d}{dx}\left(\frac{1}{2k} x^{2k}\right) =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{\cancel{2k}} \cancel{2k} \cdot x^{2k-1}$$

$$= \underbrace{\sum_{k=1}^{\infty} (-1)^k x^{2k-1}}_{} = -x + x^3 - x^5 + x^7 \ldots$$

$$= -x\,(1 - x^2 + x^4 - x^6 \ldots)$$

$$= -x \left[\sum_{k=0}^{\infty} (-1)^k x^{2k} \right] =$$

$$= -x \left[\sum_{k=0}^{\infty} (\underbrace{-x^2}_{q})^k \right]$$

$\underbrace{|-x^2| < 1}$

$$x^2 = t \qquad |x^2| < 1$$

$$\boxed{|t| < 1}$$

geometriska serie

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad (*)$$

när $|q| < 1$

$$= -x \left[\frac{1}{1 - (-x^2)} \right]$$

$$= -x \left[\frac{1}{1 + x^2} \right]$$

$$= -\frac{1}{2}\left[\frac{2x}{1 + x^2} \right] = -\frac{1}{2}\left[\frac{1}{(1+x^2)} \cdot 2x \right]$$

$$\boxed{\frac{dS(x)}{dx} = -\frac{1}{2} \frac{d}{dx}\left(\ln(1+x^2) \right)}$$

$$\Downarrow$$

$$S(x) = -\frac{1}{2}\left(\ln(1+x^2) \right) + C$$

$$S(0) = 1 + \left(\sum_{k=1}^{\infty} (-1)^k \frac{0^{2k}}{2k} \right) = 1 + 0 = C \Rightarrow \boxed{C = 1}$$

<u>Initial Condition</u>

$$\boxed{S(x) = -\frac{1}{2}\ln(1+x^2) + 1}$$ ⊛

§19.4 Differential ekvation: lösning med Potensserier:

1914a) Bestäm potensserier som satisfierar

$xy'' + y' - y = 0$ $\quad \boxed{y(0) = 1}$

Bestäm också konvergensradien.

Lösning: Antag potensserien $y(x) = \sum_{k=0}^{\infty} a_k x^k$

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 \cdot 1 + \underbrace{a_1 x^1 + a_2 x^2 + \dots}$$

$$= a_0 + \sum_{k=2}^{\infty} a_{k-1} x^{k-1} \qquad y(0) = 1 \Rightarrow \boxed{a_0 = 1}$$

$$y'(x) = \sum_{k=0}^{\infty} a_k k x^{k-1} = \cancel{0} + a_1 + \underbrace{a_2 2x^1 + a_3 3x^2 + \dots}$$

$$= a_1 + \sum_{k=2} k a_k x^{k-1}$$

$$y''(x) = 0 + \sum_{k=2} k(k-1) a_k x^{k-2} = a_2 . 2 . 1 + a_3 3.2 x^1 + \dots$$

$$\underbrace{xy''(x)} = x \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} =$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-1}$$

Ekvation: $xy'' + y' - y = 0$

$$= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-1} + \left(a_1 + \sum_{k=2}^{\infty} k a_k x^{k-1}\right) - a_0 - \sum_{k=2}^{\infty} a_{k-1} x^{k-1} = 0$$

$$= a_1 - a_0 + \sum_{k=2}^{\infty} \left(k(k-1)a_k + k a_k - a_{k-1}\right) x^{k-1} = 0$$

$$\left(a_k (k^2 - \cancel{k} + \cancel{k}) - a_{k-1}\right) x^{k-1}$$

$$= \boxed{(a_1 - a_0) + \sum_{k=2}^{\infty} \left(\underline{k^2 a_k - 1 a_{k-1}}\right) x^{k-1}} = 0 + 0x^1 + 0x^2 + \dots$$

$\left[\, a_1 - a_0 = 0 \right.$ $\quad$ Initial condition $y(0) = 1 \rightarrow \boxed{a_0 = 1}$

$$k=2$$

$$\begin{cases} a_1 - a_0 = 0 & \text{(initial condition } y(0)=1 \Rightarrow \boxed{a_0 = 1}\text{)} \\ k^2 a_k - a_{k-1} = 0 & k = 2, \dots \end{cases}$$

$$\begin{cases} a_0 = 1 \Rightarrow a_1 = a_0 = 1 \\ a_k = \underbrace{\frac{1}{k^2} a_{k-1}}, \quad k = 2, \dots \end{cases}$$

$$a_2 = \underbrace{\frac{1}{2^2}} a_1$$

$$a_3 = \frac{1}{3^2} a_2 = \underbrace{\frac{1}{3^2} \cdot \frac{1}{2^2}} a_1$$

$$a_4 = \frac{1}{4^2} a_3 = \frac{1}{4^2} \cdot \frac{1}{3^2} \cdot \frac{1}{2^2} a_1 = \frac{1}{(4.3.2.1)^2} a_1$$

$$\vdots$$

$$\boxed{a_n = \frac{1}{((n)!)^2} a_1} \qquad a_1 = 1 \;; \qquad \boxed{a_k = \frac{1}{(k!)^2}}$$

$$\boxed{y(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k} \quad \text{lösning}$$

II Bestäma konvergensradien.

Kvotformeln:
$$\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{1}{((k+1)!)^2} (k!)^2 \right| =$$

$$= \left[\frac{\cancel{k!}}{(k+1)\cancel{k!}} \right]^2 = \left[\frac{1}{(k+1)^2} \right] \xrightarrow[k \to \infty]{} 0$$

$$R = 1/k = 1/0 \to \infty \qquad I = \mathbb{R}$$

Då är $y(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k$

Konvergent för $x \in \mathbb{R}$. $\square$