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Sidan 112

Termvis Integration / Derivation av funktionsserier

X

Sidan 92

[Termvis Integration/ Derivation av Potensserier.]

Sidan 92

$\boxed{\text{Satsen } 19.6}$ Derivation och Integration av Potenserier

Antag $\sum_{k=0}^{\infty} a_k x^k$ Konvergensradien $R > 0$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{gränsfunktion} \quad -R < x < R$$

Då

$$f'(x) = \sum_{k=1}^{\infty} a_k (k) x^{k-1} = \sum_{k=1}^{\infty} k a_k x^{k-1}; \quad -R < x < R$$

och

$$\int_0^x f(t)dt = \int \sum_{k=0}^{\infty} a_k x^k dx = \sum_{k=0}^{\infty} a_k \int x^k dx = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1}$$

$$-R < x < R$$

Exempel:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}; \quad |x| < 1$$

(geometriska serien)

Enligt satsen 19.6

$$(*) \quad \underbrace{\sum_{n=1}^{n} n x^{n-1}} = 1 + 2x + \dots + n x^{n-1} + \dots \overset{(*)}{=} \frac{d}{dx}\left(\frac{1}{1-x}\right) \overset{(*)}{=} \frac{1}{(1-x)^2}$$

$$|x| < 1$$

$$(*)(*) \quad \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)} = x^1 + \frac{x^2}{2} + \frac{x^3}{3} + \dots \overset{*}{=} \int \frac{1}{1-x} dx = -\ln(1-x)$$

$$|x| < 1$$

Bestäm

$$f(x) = \underbrace{\sum_{n=0}^{\infty} n x^{n+2}}_{\text{potensserien}}$$

Lösning:

$$f(x) = \sum_{n=0}^{\infty} \underbrace{n . x^{n-1} . x^3} = x^3 \underbrace{\sum_{n=0}^{\infty} n x^{n-1}} \iff$$

$\sum^{\infty}$ $n-1$

$n=0$ $\qquad$ $n=0$

$$\frac{1}{x^3} f(x) = \underbrace{\sum_{n=0}^{\infty} n x^{n-1}}_{} \overset{*}{=} \frac{1}{(1-x)^2} \qquad |x|<1$$

derivater av geometriska serien

Då $\boxed{f(x) = \frac{x^3}{(1-x)^2} \; ; \; |x|<1}$

Ex 2: Använd Potensserien att hitta $\underbrace{\sum_{n=1}^{\infty} \frac{n}{3^n}}$

Lösning: Antag

$$\sum_{n=1}^{\infty} n \cdot \frac{1}{3^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^n = \underbrace{\sum_{n=1}^{\infty} n \cdot x^n}_{\text{potensserien.}} \quad \text{med } x = \frac{1}{3}$$

Mål: bestäm $f(x) = \sum_{n=1}^{\infty} n x^n = \sum_{n=0}^{\infty} n x^n$

$f(1/3)$.

$$f(x) = \sum_{n=0}^{n} n x^n = \sum n x^{n-1} \cdot x \iff x \neq 0$$

$$\frac{1}{x} f(x) = \underbrace{\sum_{n=0}^{\infty} n x^{n-1}}_{} = \frac{1}{(1-x)^2} \qquad |x|<1.$$

Termsvis derivater av geometriska serien

$$f(x) = \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n \qquad |x|<1.$$

$X = 1/3$ $\qquad f(1/3) = \sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^n$

$$f(1/3) = \frac{(1/3)}{(1-1/3)^2} = \frac{(1/3)}{(4/9)} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{9}}{4} = \frac{3}{4}$$

Summan är $3/4 = \sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^n$ $\square$

Ex3 Bestäm $f(x) = \sum_{n=0}^{\infty} (n+2)\, x^{n+1}$ (också konvergens intervall)

Lösning.

Tankar: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $|x|<1$

Integration

$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x)$

$|x| < 1$

Tillåta för potensserier

$$1^\circ \quad \int f(x)\,dx \overset{*}{=} \sum_{n=0}^{\infty} (n+2) \int x^{n+1}\,dx =$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \cancel{(n+2)} \frac{x^{n+2}}{\cancel{(n+2)}} + C$$

integrations konstant

$$\int f(x)\,dx = \sum_{n=0}^{\infty} x^{n+2} + C = \sum_{n=0}^{\infty} x^n \cdot x^2 + C = x^2 \sum_{n=0}^{\infty} x^n + C$$

$$\frac{1}{x^2} \int f(x)\,dx = \sum_{n=0}^{\infty} x^n + \frac{C}{x^2} \qquad |x| < 1$$

$$\frac{1}{x^2} \int f(x)\,dx = \frac{1}{1-x} + \frac{C}{x^2}$$) multiplicera allt med x^2

$$\int f(x)\,dx = \frac{x^2}{(1-x)} + C$$) deriverar

$$f(x) = \frac{d}{dx}\left(\frac{x^2}{(1-x)}\right) + 0$$

$$f(x) = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2}{(1-x)^2} =$$

$$f(x) = \frac{2x - x^2}{(1-x)^2} \quad ; \quad |x| < 1$$

Konvergensintervallet.

Ex 4. Bestäm $\alpha = \sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n} = \sum_{n=0}^{\infty} (n+1)^2 \underbrace{\left(\frac{1}{\pi}\right)^n}_{x^n}$

Lösning:

När vi tänker på potensserier:

fråga är: Hitta $f(x) = \sum_{n=0}^{\infty} (n+1)^2 (x)^n$ $\quad x = \frac{1}{\pi} \in [-1, 1]$

$\alpha = f(1/\pi)$

$$\int f(x)dx = \sum_{n=0}^{\infty} (n+1)^2 \int (x)^n dx + C$$

$$\int f(x)dx = \sum_{n=0}^{\infty} (n+1)^{\cancel{2}} \frac{x^{n+1}}{\cancel{(n+1)}} + C = \sum_{n=0}^{\infty} (n+1)\, x^{n+1} + C$$

$x^{n+1} = x^n \cdot x$

$(x \neq 0)$

$$\frac{1}{x}\int f(x)dx = \sum_{n=0}^{\infty} (n+1)\, x^n + \frac{C}{x}$$

$$\int\left(\frac{1}{x}\int f(x)\,dx\right)dx \overset{*}{=} \sum_{n=0}^{\infty} \cancel{(n+1)} \frac{x^{n+1}}{\cancel{(n+1)}} + \int \frac{C}{x}dx + D$$

$$\int\left(\frac{1}{x}\int f(x)\,dx\right)dx = \sum_{n=0}^{\infty} x^n \cdot x + \int \frac{C}{x}dx + D$$

$$= x \sum_{n=0}^{\infty} x^n \quad \text{geom serien}$$

$$\int \frac{1}{x}\int f(x)\,dx\,dx = x \cdot \frac{1}{(1-x)} + \int \frac{C}{x}dx + D \quad \text{Derivera}$$

$$\frac{1}{x}\int f(x)dx = \frac{d}{dx}\left[x \cdot \frac{1}{(1-x)}\right] + \frac{C}{x} + 0$$

$$\frac{1}{x}\int f(x)\,dx = \frac{\cancel{(1-x)} - \cancel{x(-1)}}{(1-x)^2} + \frac{C}{x} \quad (**)$$

$$\int f(x)\,dx = \frac{x}{(1-x)^2} + \frac{C \cdot \cancel{x}}{\cancel{x}}$$

derivera r

$$\boxed{f(x) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) + 0}$$

$$f(1/\pi) = \alpha$$