Ö13_V7_12maj2021

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Uppgifter: 1930 / 1932 / 1938

Sats 19.10 f_n funktionsföljd.

$$\left[\begin{array}{l} f_n \to f \text{ likformigt på } [a,b] \\ f_n \text{ kontinuerliga på } [a,b] \end{array}\right] \Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx \overset{\circledast}{=} \int_a^b \underbrace{\lim_{n\to\infty} f_n(x)}\,dx \overset{\circledast}{=} \int_a^b f(x)\,dx$$

Sats 19.11 $I = [a,b]$

$$\lim_{n\to\infty} \int_I f_n(x)\,dx \overset{\circledast}{=} \int_I \lim_{n\to\infty} f_n(x)\,dx \overset{\circledast}{=} \int_I f(x)\,dx$$ gäller under

förutsättningar:

1) Integralen $\boxed{\int_I f_n(x)\,dx}$ existerar $\forall n \geq n_0$

2) $f_n \to f$ punktvis på I och $\int_I f(x)\,dx$ existerar

3) $\exists\ g: I \to \mathbb{R}$
$x \to g(x)$ $\quad |f_n(x)| \leq g(x) \quad \forall x \in I, \forall n \geq n_0$.

och $\int_I g(x)\,dx$ existerar.

Sidan 112

1930 Låt $\underbrace{f_n(x) = \sqrt{x}\, e^{-x/n}}$. Beräkna $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$

Lösning:

$I = [a,b] = [0,1]$

$x = 0 \Rightarrow f_n(0) = \sqrt{0}\, e^{-0/n} = 0 \frac{1}{e^{0/n}} = 0 \cdot 1 = 0 \quad \forall n \in \mathbb{N}$.

$x = 1 \Rightarrow f_n(1) = \sqrt{1}\, e^{-1/n} = 1 \frac{1}{e^{1/n}} = 1 \frac{1}{\sqrt[n]{e}} \xrightarrow[n\to\infty]{} 1$

$(\sqrt[n]{a} \underset{n\to\infty}{=} 1)$

$0 < x < 1 \quad f_n(x) = \sqrt{x}\, e^{-x/n} = \sqrt{x} \frac{1}{e^{x/n}} = \sqrt{x} \frac{1}{\sqrt[n]{e^x}} \xrightarrow[n\to\infty]{} \sqrt{x}$

$f_n(x) = \sqrt{x}\, e^{-x/n} \xrightarrow{} f(x) = \sqrt{x}, \quad \forall x \in [0,1]$

punktvis Konvergerar $\circledast\circledast$

Kan vi också bevisa att $f_n \to f$ likformigt ?

Låt $\underline{\varepsilon > 0}$ (vi behöver hitta $\underline{N_\varepsilon}$ så att $n > N_\varepsilon \Rightarrow |f_n - f| < \varepsilon$)

$$|f_n(x) - f(x)| \overset{(**)}{=} \left| \sqrt{x}\, e^{-x/n} - \sqrt{x} \right| = \sqrt{x}\, |e^{-x/n} - 1| = \sqrt{x}\left(1 - \frac{1}{e^{x/n}}\right) <$$

$e^{-x/n}$ = avtagande

$$\left.\begin{array}{l} 1 - e^{-x/n} = \text{växande} \\ \sqrt{x} = \text{växande} \end{array}\right\} \text{strängt} \Longrightarrow \underline{\text{Strängt växande}} \text{ på } I = [0, 1]$$

$x = 1$

$\forall n \in \mathbb{N}.$

Då har vi

$$|f_n(x) - f(x)| = \sqrt{x}\, |1 - e^{-x/n}| \leq \underbrace{\sqrt{1}\left(1 - \frac{1}{e^{1/n}}\right)}_{\text{Talföljd som är konvergent}} \quad \forall x \in [0,1] \quad \forall n \in \mathbb{N}$$

$$a_n = \left(1 - \frac{1}{\sqrt[n]{e}}\right) < 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - \frac{1}{\sqrt[n]{e}} = 1 - 1 = 0$$

$\underline{0 < \varepsilon < 1}$ $\quad |a_n| = \underbrace{\left|1 - \frac{1}{e^{1/n}}\right| < \varepsilon}$ < 1

$$\underbrace{1 - \varepsilon} < \frac{1}{e^{1/n}} = \underbrace{e^{-1/n}}$$

$$\ln(1-\varepsilon) < \ln(e^{-1/n}) = -\frac{1}{n}\ln(e) = -\frac{1}{n}$$

$$-\ln(1-\varepsilon) > \frac{1}{n} \iff$$

$$n > \frac{-1}{\ln(1-\varepsilon)} = N_\varepsilon$$

Då $n > N_\varepsilon \Rightarrow$

$$\boxed{|f_n(x) - f(x)| < \sqrt{x}(1 - e^{-x/n}) < |a_n - 0| < \varepsilon}$$

$x = 1$

Då $f_n \to f$ likformigt på $\underline{I = [0, 1]}$

Enligt Sats 19.10

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx =$$

$$= \int_0^1 \sqrt{x}\,dx = \left.\frac{x^{\frac12+1}}{\frac12+1}\right|_0^1 = \frac{(1)^{3/2}}{3/2} = \frac{2}{3} \quad \square$$

1932 Beräkna: $\displaystyle\lim_{n\to\infty}\int_2^\infty \underbrace{n^{-1}\cdot x^{-2}\cdot \ln(x^n-1)}\,dx$

Lösning: $I = [2,\infty)$

$$f_n = n^{-1}\cdot x^{-2}\ln(x^n-1) = \frac{1}{n}\frac{1}{x^2}\ln(x^n-1) \qquad \forall x\in[2,\infty)$$

$$f_n(2) = \frac{1}{n}\frac{1}{2^2}\ln(2^n-1) < \frac{1}{n}\frac{1}{2^2}\ln(2^n) = \frac{1}{\cancel{n}}\frac{1}{2^n}\cancel{n}\ln(2)$$

$$f_n(e) = \frac{1}{n}\frac{1}{e^2}\ln(e^n-1) < \frac{1}{n}\frac{1}{e^2}\ln(e^n) = \frac{1}{\cancel{n}}\frac{1}{e^2}\cancel{n}\,\overset{1}{\cancel{\ln(e)}} = \frac{1}{e^2}$$

$$f_n(x) = \frac{1}{n}\frac{1}{x^2}\ln(x^n-1) < \frac{1}{n}\frac{1}{x^2}\ln(x^n) = \frac{1}{\cancel{n}}\cdot\frac{1}{x^2}\cancel{n}\ln(x) = \frac{\ln(x)}{x^2}$$

$$f_n(x) = \frac{1}{n}\frac{1}{x^2}\ln(x^n-1) < \underbrace{\frac{\ln(x)}{x^2} = g(x)}, \quad x\in[2,+\infty)$$

Sat 19.11

3° Nu hittade vi $g(x) = \dfrac{\ln(x)}{x^2}$ ~~som~~

$$|f_n(x)| \le g(x) \qquad \forall x\in[2,\infty)$$

$$\underbrace{\int_2^\infty g(x)\,dx} = \int_2^\infty \frac{\ln x}{x^2}\,dx \overset{\text{Del integration:}}{=\!=}$$

$u = \ln x \to du = 1/x$

$dv = x^{-2} \to v = -x^{-1}$

$$= \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_2^\infty = \left[\frac{1}{x}\left(\ln(x)+1\right)\right]_\infty^2 =$$

$$= \frac{1}{2}\left[\ln(2)+1\right] \qquad \int_2^\infty g(x)\,dx \text{ existerar}$$

$$= \frac{1}{2}\left[\ln(2)+1\right] \qquad \int_2^{\infty} g(x)\,dx \text{ existerar}$$

$g(x)$ är våra Majorantfunktion $\forall x \in [2,\infty)$
$\forall n \in \mathbb{N}$.

2° ? $f_n \to f$? vilken f ?
punktvis

$$f_n(x) = \frac{1}{n}\frac{1}{x^2}\ln(x^n-1) = \frac{1}{n}\frac{1}{x^2}\ln\left(x^n\left(1-\frac{1}{x^n}\right)\right) =$$

$$f_n(x) = \frac{1}{n}\frac{1}{x^2}\left[\ln(x^n) + \ln\left(1-\frac{1}{x^n}\right)\right] =$$

$$f_n(x) = \frac{1}{n}\frac{1}{x^2}\left[n\ln(x) + \ln\left(1-\frac{1}{x^n}\right)\right] =$$

$$f_n(x) = \frac{\ln(x)}{x^2} + \frac{1}{n}\frac{1}{x^2}\ln\left(1-\frac{1}{x^n}\right)$$

$$\lim_{n\to\infty} f_n(x) = \frac{\ln(x)}{x^2} + \lim_{n\to\infty}\underbrace{\frac{1}{n}}_{\to 0}\cdot\frac{1}{x^2}\cancel{\ln\Big(\underbrace{1-\overbrace{\frac{1}{x^n}}^{\to 0}}_{1}\Big)} =$$

$$\lim_{n\to\infty} f_n(x) = \frac{\ln(x)}{x^2} = f(x) \text{ gränsfunktion} \qquad \boxed{g(x) = f(x)}$$

$$\forall x \in [2,\infty)$$

och $\int_I f(x)\,dx = \int_I g(x)\,dx$ existerar.

Enligt satsen 19.11

$$\lim_{n\to\infty}\int_I f_n(x)\,dx \overset{(*)}{=} \int_I \lim_{n\to\infty} f_n(x)\,dx = \int_I f(x)\,dx =$$

$$= \frac{1}{2}\left(1+\ln(2)\right)$$

□

Sats 19.13 Termvisintegration vid Dominerad Konvergens

$$\int_I \sum_{k=1}^{\infty} u_k(x)\,dx \overset{(*)}{=} \sum_{k=1}^{\infty}\int_I u_k(x)\,dx \quad \text{gäller under}$$

förutsättningar.

1) $\int_I u_k(x)\,dx$ existerar för alla k

2) $\sum_{k=1}^{\infty} u_k(x)$ konvergerar $\forall x \in I$.

$= S(x)$

$\int_J S(x)\,dx$ existerar

3°) $\exists\ g(x)$: $g: I \to \mathbb{R}$ så att

$$\left|\sum_{k=1}^{n} u_k(x)\right| \leq g(x) \quad \forall x \in I,\ \forall n \in \mathbb{N}$$

$\int_I g(x)\,dx$ existerar

Sats 19.14 Om termvis derivation.

Antag att
- 1) $\sum_{k=1}^{\infty} u_k(x)$ konvergerar punktvis mot $S(x)$ på I.
- 2°) $u_k'(x)$ kontinuerliga $\forall k \in \mathbb{N}$.
- 3°) $\sum_{k=1}^{\infty} u_k'(x) \longrightarrow g(x)$ likformigt på I

Då är $\boxed{S'(x) = g(x)}$

$$\frac{d}{dx}\left(\sum_{k=1}^{\infty} u_k(x)\right) = \sum_{k=1}^{\infty} \frac{d}{dx}\left(u_k(x)\right) \quad x \in I.$$

1938 Bevisa att

$$\int_0^1 \frac{\ln(x)}{(x-1)}\,dx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{k^2} \dots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Lösning

Mål Använda potensserier

Lösning

$$f(x) = \frac{\ln(x)}{(x-1)}$$

Mål Använda potensserier

variabel byte: $(x-1) = t$ $\quad x = 0 \Rightarrow t = -1$

$x = t+1$ $\quad x = 1 \Rightarrow t = 0$

$$\int_0^1 f(x)\,dx = \int_{-1}^{0} \frac{\ln(1+t)}{t}\,dt$$

Men nu för $\ln(1+t)$ känner vi potensserier

$$\ln(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k \quad ; \quad |t| < 1$$

$$\frac{\ln(1+t)}{t} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot t^{k-1} \qquad |t| < 1 \qquad (-1, 1)$$

Termvis integration

$$\int_{-1}^{0} \frac{\ln(1+t)}{t}\,dt \overset{*}{=} \int \sum \overset{*}{=} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_{-1}^{0} t^{k-1}\,dt =$$

Enligt sats 19.13 (19.6) potensserier)

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[\frac{t^k}{k} \right]_{-1}^{0} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \left[0^k - (-1)^k \right] =$$

$$\ominus \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (-1)^k}{k^2} =$$

$k=1 \quad (-1)^0 (-1)^1 = -$

$k=2 \quad (-1)^1 (-1)^2 = -$

$$= \boxed{\sum_{k=1}^{\infty} \frac{1}{k^2}} \qquad \forall x \in I = [0,1]$$

Extra $\quad$ Euler's bevis för beräkna $\sum_{k=1}^{\infty} \frac{1}{k^2}$

Eulers lösning. Taylor utveckling av

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7} \ldots$$

$$\boxed{\sin(x) = 0} \qquad x = 0, \pi, -\pi, +2\pi, -2\pi, \ldots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \overset{?}{=} C\,(x-0)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\dots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = C\,(\cancel{x})(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\dots$$

$$\boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots = C\,(x^2-\pi^2)(x^2-4\pi^2)\dots}$$

Att bestämma C

$\boxed{x=0} \Longrightarrow \quad 1 - 0 + 0 \dots = C\,(-\pi^2)(-4\pi^2)(-9\pi^2)\dots$

$$\Rightarrow \quad \boxed{C = \frac{1}{(-\pi)^2\,(-4\pi^2)\,(-9\pi^2)\dots}}$$

Då

$$1 - \frac{1}{3!}x^2 + \frac{x^4}{5!} + \dots = \frac{(x^2-\pi^2)(x^2-4\pi^2)\dots}{(-\pi)^2\;(-4\pi)^2\;\dots}$$

$$1 - \frac{1}{3!}x^2 + \frac{x^4}{5!} + \dots = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\dots$$

$$-\frac{1}{3!}x^2 \longleftrightarrow -\frac{x^2}{\pi^2} - \frac{x^2}{4\pi^2} - \frac{x^2}{9\pi^2} - \dots$$

$$\cancel{-}\frac{1}{3!}\cancel{x^2} \longrightarrow \cancel{-}\frac{\cancel{x^2}}{\pi^2}\left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots\right)$$

$$\boxed{\frac{\pi^2}{3!} = \sum_{k=1}^{\infty} \frac{1}{k^2}}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$$