L13_V8_18maj

den 18 maj 2021 07:42

Mål
- Uppgift om: Differentialekvationslösning med potensserie
- Uppgift om funktionsserie (med sats 19.11)

I. a) Bestäm i form av en potensserie en lösning till Differentialekvationen:

$$\begin{cases} (4-x^2)y'' + y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

b) Bestäm seriens konvergensradie.

Lösning: Antag $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \leftrightarrow \sum_{k=0} (k+1) a_{k+1} x^k \leftrightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$n-1=k$
$n=k+1$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \leftrightarrow \sum_{k=0} (k+2)(k+1) a_{k+2} x^k \leftrightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$n-2=k$
$n=k+2$
$n-1=k+1$

$$(4-x^2)y'' + y = 0$$

$$x^2 y'' = x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2} n(n-1) a_n x^n = \sum_{n=0} n(n-1) a_n x^n$$

$$4y'' = 4 \sum_{n=0} (n+2)(n+1) a_{n+2} x^n = \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n$$

Då

$$0 = (4-x^2)y'' + y = \sum_{n=0}^{\infty} \underbrace{\left(4(n+2)(n+1)a_{n+2} - n(n-1)a_n + a_n\right)}x^n \Longleftrightarrow$$

$$4(n+2)(n+1) a_{n+2} - a_n(n(n-1) - 1) = 0$$

$$4(n+2)(n+1) a_{n+2} = (n^2 - n - 1) a_n$$

$$\Big(a_{n+2} = (n^2-n-1) a_n \qquad n = 0, 1, 2, \dots \quad n \in \mathbb{N}$$

$$\begin{cases} a_{n+2} = \dfrac{(n^2-n-1)\,a_n}{4(n+2)(n+1)} & n=0,1,2,\ldots \quad n\in\mathbb{N} \\ \text{Med} \quad y(0)=1 \\ \qquad\quad y'(0)=0 \end{cases}$$

$$y(0)=1 \Rightarrow y(x)\Big|_{x=0} = \sum_{n=0}^{\infty} a_n x^n \Big|_{x=0} = a_0 + a_1\cancel{x} + a_2\cancel{x^2} + \ldots \Rightarrow \boxed{a_0 = 1}$$

$$y'(0)=0 \Rightarrow y'(x)\Big|_{x=0} = \sum_{n=1} n a_n x^{n-1}\Big|_{x=0} = a_1 + 2a_2\cancel{x} + 3a_3\cancel{x^2} + \ldots \Rightarrow \boxed{a_1 = 0}$$

Då har vi
$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_{n+2} = \dfrac{n^2-n-1}{4(n+2)(n+1)}\, a_n \end{cases}$$

$$n=0 \Rightarrow \quad a_2 = \frac{-1}{4(2)(1)}\, a_0 = -\frac{1}{8}\, a_0 = -\frac{1}{8} \qquad (a_0 = 1)$$

$$n=1 \Rightarrow \quad a_3 = \frac{\cancel{(1)^2} - \cancel{1} - 1}{4(3)(2)} \cdot a_1 = -\frac{1}{24} \cdot a_1 = 0 \qquad (a_1 = 0)$$

$$n=2 \Rightarrow \quad a_4 = \frac{2^2-2-1}{4(4)(3)}\, a_2 = \frac{1}{16*3}\left(-\frac{1}{8}\right) = -\frac{1}{384}$$

$$n=3 \Rightarrow \quad a_5 = \frac{(3)^2-3-1}{4(5)(4)}\, a_3 = 0 \Rightarrow a_7 = 0 \ldots \Rightarrow a_{2n-1} = 0 \quad ; n \geq 1.$$

$\vdots$

Obs: $\quad a_{n+2} = \dfrac{n^2-n-1}{4(n+2)(n+1)}\, a_n \quad n\geq 0 \quad$ (utan enkelt uttryckt i "sluten form")

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$
$$= 1 \quad -\frac{1}{8}x^2 \quad -\frac{1}{384}x^4 \quad \ldots$$

I.b. Att bestämma seriens konvergensradie:

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$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \begin{cases} a_{2n} \neq 0 \\ a_{2n+1} = 0 \end{cases}$$

$$y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} \qquad x^2 = t \Rightarrow \sum_{k=0}^{\infty} a_{2k} t^k \quad \circledast$$

Då Enligt satsen 19.5 bestäms $R^* =$ konvergensradie för denna serie $\circledast$

$$K = \lim_{k\to\infty} \left| \frac{a_{2(k+1)}}{a_{2k}} \right|$$

$$a_{n+2} = \frac{n^2 - n - 1}{4(n+2)(n+1)} a_n \qquad a_{2k+2} \underset{n=2k}{=} \frac{(2k)^2 - (2k) - 1}{4(2k+2)(2k+1)} a_{2k}$$

$$K = \lim_{k\to\infty} \left| \frac{a_{2k+2}}{a_{2k}} \right| = \lim_{k\to\infty} \left| \frac{\frac{(2k)^2 - (2k) - 1}{4(2k+2)(2k+1)} \cancel{a_{2k}}}{\cancel{a_{2k}}} \right| =$$

$$= \lim_{k\to\infty} \frac{\cancel{4k^2}\left(1 - \frac{1}{2k} - \frac{1}{4k^2}\right)}{4 \cdot \cancel{2k}\left(1 + \frac{2}{2k}\right)\cancel{2k}\left(1 + \frac{1}{2k}\right)} = \frac{1}{4}$$

Då är $R^* = \frac{1}{K} = 4$. $\therefore$ serien är konvergent då $|t| < 4$

$$t = x^2 \Rightarrow |t| < 4 \Leftrightarrow |x^2| < 4 \Leftrightarrow |x| < 2$$

Då
$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{1}{8}x^2 - \frac{1}{384}x^4 - \dots \quad ; \quad |x| < 2}$$

II. Funktionsserier:

Sats 19.11 (om dominerad konvergens)

Tes
$$\boxed{\lim_{n\to\infty} \int_I f_n(x)\,dx = \int_I f(x)\,dx}$$

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Om: 1) $\int_I f_n(x)\,dx$ existerar $\forall\, n \geq n_0$.

Hipotes

2) $f_n \to f$ punktvis på I och $\int_I f(x)\,dx$ existerar

3) $\exists\; g: I \to \mathbb{R}$, $\int_I g(x)\,dx$ existerar och $|f_n(x)| \leq g(x)$ $\forall\, x \in I$ $\forall\, n \geq n_0$.

Uppgift 1936* sidan 112.

Visa att a) $$\int_0^n \left(1-\frac{x}{n}\right)^n \ln x\,dx = \frac{n}{(n+1)}\left(\ln n - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots - \frac{1}{n+1}\right)$$

b) $$\int_0^\infty e^{-x} \ln x\,dx = -\gamma \equiv \text{Eulers konstant.}$$

a) <u>Lösning</u>.

$$\int_0^n \left(1-\frac{x}{n}\right)^n \ln x\,dx$$

Variabel byte

$t = 1 - \frac{x}{n}$ $\quad x=0 \Rightarrow t=1$ $\quad x=n \Rightarrow t=0$ $\quad dt = -\frac{1}{n}\,dx \therefore dx = -n\,dt$

$-x = n(t-1)$

$x = n(1-t)$

$\Downarrow$

$$\int_1^0 t^n \ln(n(1-t)).(-n)\,dt = \int_0^1 n\,t^n \ln(n(1-t))\,dt$$

<u>2 alternativer</u>

1) Partial integrering med tryck

2) serieutveckling till $\ln(1-t)$

Alternativ (1) Partial integrering med Tryck!

$$\int_0^1 \underbrace{nt^n}_{dv} \underbrace{\ln(n(1-t))}_{u}\,dt = \underbrace{[u.v]_0^1}_{(A)} - \underbrace{\int_0^1 v.\,du}_{(B)}$$

$u = \ln(n(1-t)) \longrightarrow du = \frac{1}{n(1-t)}.(-n) = \frac{1}{t-1}$

$dv = nt^n \underset{\text{TRYCK!}}{\Longrightarrow} v = \frac{n(t^{n-1} - 1)}{(n+1)}$

Man kan välja en konstant här, därför att dv är fortfarande nt^n.

(A) $[uv]_0^1 = \left[\ln(n(1-t)).\frac{n(t^{n-1}-1)}{n+1}\right]_{t=0}^{t=1}$

(A) $[uv]_0^1 = \left[\ln(n(1-t)) \cdot \frac{n(t^{n+1}-1)}{n+1}\right]_{t=0}^{1}$

$$\ln(n.0).n.\frac{0}{n+1} - \ln(n).\frac{n}{n+1}(-1) = \frac{n}{(n+1)}\ln(n)$$

(B) $-\int v\,du = -\int_0^1 \frac{1}{(t-1)} \cdot \frac{n}{n+1}(t^{n+1}-1)\,dt =$

$$-\frac{n}{n+1}\int_0^1 \underbrace{\frac{t^{n+1}-1}{t-1}}_{*}\,dt = \frac{-n}{n+1}\int_0^1 \sum_{k=0}^{n} t^k\,dt = \frac{-n}{n+1}\sum_{k=0}^{n}\int_0^1 t^k\,dt$$

$$-\frac{n}{n+1}\sum_{k=0}^{n}\left[\frac{t^{k+1}}{k+1}\right]_{t=0}^{t=1} = -\frac{n}{n+1}\sum_{k=0}^{n}\left[\frac{1}{k+1} - 0\right]$$

Då

$$\int_0^n \left(1-\frac{x}{n}\right)^n \ln x\,dx = \underbrace{\frac{n}{n+1}\ln(n)}_{A} - \underbrace{\frac{n}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}}_{B}$$

$$= \frac{n}{n+1}\left[\ln n - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n+1}\right]$$

∎

Alternativ 2.

$\ln(a\cdot b) = \ln(a) + \ln(b)$

$$\int_0^1 nt^n \ln(n\cdot(1-t))\,dt = \int_0^1 nt^n\left[\ln(n) + \ln(1-t)\right]dt =$$

$$= \underbrace{\int_0^1 n\ln n\, t^n\,dt}_{(A)} + \underbrace{\int_0^1 n t^n \ln(1-t)\,dt}_{(B)}$$

(A) $n\ln n \int_0^1 t^n\,dt = n\ln n\left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{t=1} = \frac{n}{n+1}\ln(n)$

(B) $n\int_0^1 t^n \ln(1-t)\,dt$

$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \dots -$

$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} \dots$

$\int_0$ $\qquad\qquad \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} \dots$

$$n\int_0^1 t^n\left[-t-\frac{t^2}{2}-\frac{t^3}{3}-\dots\right]dt = n\int_0^1 -t^{n+1} - \frac{t^{n+2}}{2} - \frac{t^{n+3}}{3} - \dots \, dt =$$

$$-n\left[\frac{t^{n+2}}{n+2} + \frac{t^{n+3}}{n+3}\cdot\frac{1}{2} + \frac{t^{n+4}}{n+4}\cdot\frac{1}{3} + \dots\right]_{t=0}^{t=1} =$$

$$-n\left[\frac{1}{n+2}\cdot\frac{1}{1} + \frac{1}{n+3}\cdot\frac{1}{2} + \frac{1}{n+4}\cdot\frac{1}{3} + \frac{1}{n+5}\cdot\frac{1}{4} + \dots\right] =$$

$$-n\sum_{k=1}^{\infty}\frac{1}{(n+1+k)}\cdot\frac{1}{k} = -n\sum_{k=1}^{\infty}\left(\frac{A}{(n+1+k)} + \frac{B}{k}\right) \stackrel{*}{=}$$

$$\frac{A.k + B(n+1+k)}{(n+1+k)\cdot k} = \frac{1}{(\quad)k} \iff$$

$$(A+B)k = 0 \Rightarrow A = -B \Rightarrow A = -\frac{1}{(n+1)}$$
$$B(n+1) = 1 \Rightarrow B = \frac{1}{n+1}$$

$$= -n\cdot\frac{1}{n+1}\sum_{k=1}^{\infty}\left(\frac{-1}{(n+1+k)} + \frac{1}{k}\right)$$ Teleskop sum

$$= -\frac{n}{n+1}\left(\frac{-1}{n+1+1} + \frac{1}{1} - \frac{1}{n+3} + \frac{1}{2} - \frac{1}{n+4} + \frac{1}{3} - \dots - \frac{1}{(n+1)+(n+2)} + \frac{1}{n+1} - \frac{1}{(n+1)+(n+3)} + \frac{1}{n+2}\dots\right)$$
$= 0$

$$= -\frac{n}{n+1}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right)$$

Då $$\int_0^1 nt^n \ln(n(1-t))\,dt = n\frac{\ln(n)}{n+1} - \frac{n}{n+1}\left(1 + \frac{1}{2} + \dots + \frac{1}{n+1}\right)$$

$$= \frac{n}{n+1}\left(\ln(n) - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n+1}\right)$$

$\boxtimes$

1936 (b) $\displaystyle\int_0^{\infty} e^{-x}\ln x\,dx = -\gamma$

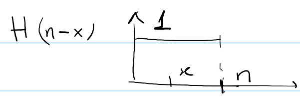
Här γ = Eulers konstant (sidan 70 upp. 1825)

$S_n = \sum_{k}^{n} \frac{1}{k}$

$$\begin{cases} S_n = \sum_{k=1}^{n} \frac{1}{k} \\ S_n - \ln(n) = \gamma + \varepsilon_n, \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0 \end{cases}$$

Enligt(a)

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \ln(x)\, dx = \lim_{n\to\infty} \frac{n}{n+1} \left(\ln(n) - \sum_{k=0}^{n} \frac{1}{k+1} \right)$$



Lösning:

$$f_n(x) = H(n-x) \left(1 - \frac{x}{n}\right)^n \ln(x) \qquad H(n-x) = \begin{cases} 1 & (n-x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

- $f_n(x) \xrightarrow[n\to\infty]{} e^{-x} \ln x = f(x)$ pointwise $\qquad n - x > 0$, $n > x$
- $\int_0^\infty f_n(x)\, dx = S_n$ enligt (a)
- $f_n(x) \leq e^{-x} \ln(x) = g(x)$ $(= f(x))$

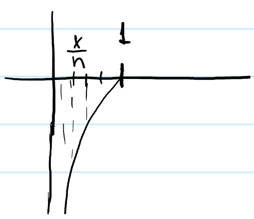
eftersom:

$$H(n-x)\left(1 - \frac{x}{n}\right)^n \cancel{\ln x} \leq e^{-x} \cancel{\ln(x)} \qquad \forall x \in (0, n] \; ?$$

$$1. \left(1 - \frac{x}{n}\right)^n \leq e^{-x} \qquad \forall x \in (0, 1]$$

$$e^{\cancel{n} \ln\left(1 - \frac{x}{n}\right)} \leq e^{\cancel{n}\left(-\frac{x}{n}\right)} = e^{-x} \iff$$

$$\ln\left(1 - \frac{x}{n}\right) \leq \left(-\frac{x}{n}\right) \quad \checkmark$$



$\frac{x}{n} \to 1$

$-\frac{x}{n} \to -1$

$\ln\left(1 - \frac{x}{n}\right) \to -\infty$

Då satisfieras

alla 3 förutsättningar av satsen 19.11.

Då

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \ln x\, dx = \lim_{n\to\infty} \int_0^\infty \underbrace{H(n-x)\left(1 - \frac{x}{n}\right)^n \ln x}_{f_n(x)}\, dx =$$

$$= \int_0^\infty e^{-x} \ln x\, dx = \lim_{n\to\infty} \frac{n}{n+1} \left[\ln(n) - \left(\sum_{k=0}^{n} \frac{1}{k+1} \right) \right]$$

$$= -\lim_{n\to\infty} \frac{n}{n+1}\left[\sum_{k=0}^{n} \frac{1}{k+1} - \ln(n)\right]$$

$$= -1 \cdot \gamma \cancel{-\varepsilon_n}, \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0$$

$$= -1 \cdot \gamma .$$