Ö14_V8_18maj2021

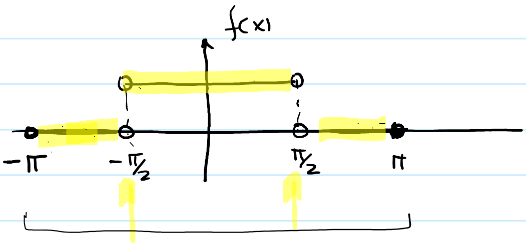
den 18 maj 2021 13:44

Mål ELW 2003 a), 2006 a), 2007

ELW 2003 a) (sidan 126 - Del 3)

Bestäm fourierserie till $f(x)$

$$f(x) = \begin{cases} 1, & \text{då } |x| < \pi/2 \\ 0, & \text{då } \frac{\pi}{2} < |x| \leq \pi \end{cases}$$



Lösning.

$$f(x) \sim s(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \qquad M_n = \{1, \cos kx, \sin kx\} \quad k = 1 \ldots, n$$

$a_0 = \langle f(x), 1 \rangle$

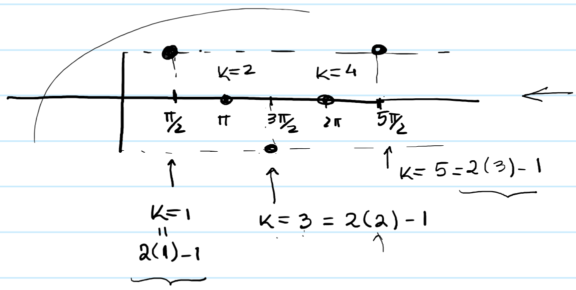
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot 1\, dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot dx = \frac{1}{\pi}\left[x\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{\pi}\cdot\pi = 1.$$

$$a_k = \langle f(x), \cos kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot \cos kx\, dx =$$

$$\frac{1}{\pi}\left[\frac{\sin kx}{k}\right]_{x=-\pi/2}^{x=\pi/2} = \frac{1}{\pi}\cdot\frac{1}{k}\left[\sin\left(k\cdot\frac{\pi}{2}\right) \overbrace{- \sin\left(k\cdot\left(-\frac{\pi}{2}\right)\right)}^{+}\right] =$$

$$\frac{1}{\pi}\cdot\frac{1}{k}\left[2 \cdot \sin\left(k\cdot\frac{\pi}{2}\right)\right]$$

$$\boxed{a_k = \frac{2}{\pi}\,\frac{1}{(2k-1)}(-1)^{k-1}}$$



$k = 1 \quad \sin\frac{\pi}{2}\cdot 1 = 1 \qquad 2k-1,\ k=1$

$k = 2 \quad \sin\frac{\pi}{2}\cdot 2 = 0$

$k = 3 \quad \sin\frac{\pi}{2}\cdot 3 = -1 \qquad 2k-1;\ k=2$

$$b_k = \langle f(x), \sin kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot \sin(kx)\, dx =$$

$$= -\frac{1}{\pi}\left[\frac{\cos(kx)}{k}\right]_{-\pi/2}^{\pi/2} = -\frac{1}{\pi}\cdot\frac{1}{k}\left[\underbrace{\cancel{\cos\left(k\frac{\pi}{2}\right) - \cos\left(-\frac{k\pi}{2}\right)}}_{= 0\ \forall k \in \mathbb{N}}\right]$$

jämt funktion.

$$\boxed{b_k = 0} \quad \forall k \in \mathbb{N}$$

$-1/2$

$= \forall k \in \mathbb{N}$.

jämt funktion.

$$\boxed{b_k = 0} \quad \forall k \in \mathbb{N}$$

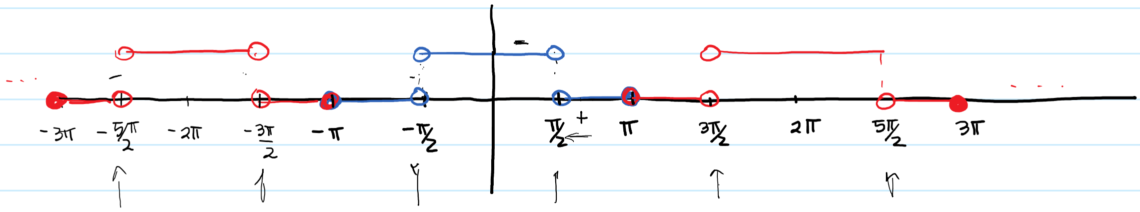
Då $$f(x) \sim \underbrace{\frac{1}{2}}_{a_0} + \frac{2}{\pi} \sum_{k=1}^{\infty} \underbrace{\frac{(-1)^{k-1}}{(2k-1)}}_{a_k} \cos((2k-1)x)$$ ∎

2006 a) Bestäm för varje reellt tal x summan av Fourier serie till funktion $f(x)$ (2003 a)

Lösning :

För att betrakta $x \in \mathbb{R}$, utökar vi definition av $f(x)$ genom att kräva 2π- periodicitet

$$f(x+2\pi) = f(x) \quad \Leftarrow \text{Periodisk fortsättning}$$



$$f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)} \cos((2k-1)x) = S(x) \quad , \quad x \in [-\pi, \pi]$$

f deriverbar / kontinuerlig för $\boxed{x \neq \pm \frac{\pi}{2} + 2\pi(n), \ n \in \mathbb{Z}.}$

Sats 20.2

Då $$\boxed{S(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] \qquad x = \pm \pi/2 + 2\pi(n)}$$

$$= \frac{1}{2}\left[0 + 1\right] \qquad x = \pi/2 + 2\pi n \quad n \in \mathbb{N}$$

$$= \frac{1}{2}\left[1 + 0\right] \qquad x = -\pi/2 - 2\pi n \quad n \in \mathbb{N}$$

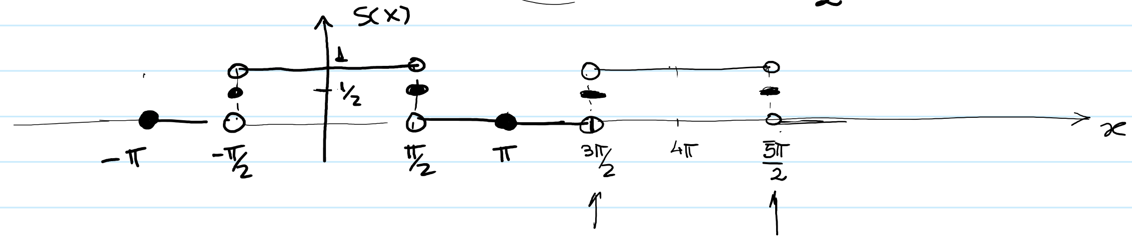
$$\boxed{S(x) = 1/2} \quad x = \pm \pi/2 + 2\pi . n \ , \ n \in \mathbb{Z}.$$

Då $S(x)$ = summan av Fourierserien är

$$S(x) = \begin{cases} f(x) & \forall x \text{ där } f \text{ är kontinuerlig} \\ 1/2 & x = \pm \frac{\pi}{2} + 2\pi . n \ , \end{cases} \qquad \text{Sats 20.2}$$

$S(x)$

$\frac{1}{2}$ $\quad x = \pm\frac{\pi}{2} + 2\pi \cdot n$, $\quad$ Sats 20.2



Uppgift 2007 sidan 132.

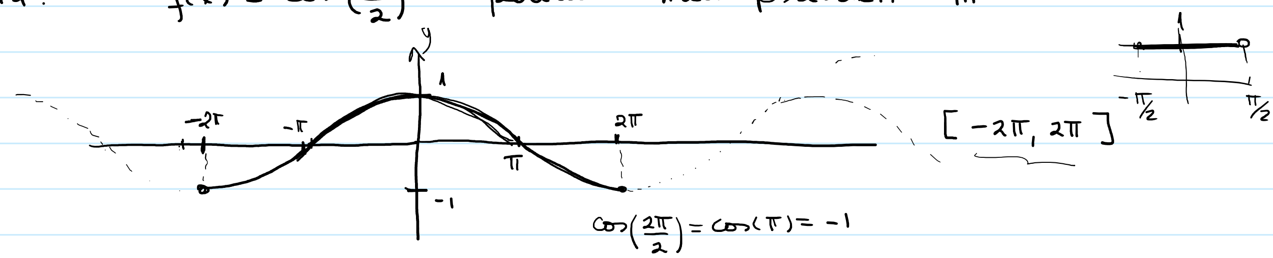
a) Bestäm Fourierserien till $\boxed{f(x) = \cos\left(\frac{x}{2}\right),\ x \in \mathbb{R}}$

b) Ange FS summan

c) Beräkna $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 - \frac{1}{4}}$

Lösning.

2007a. $\quad f(x) = \cos\left(\frac{x}{2}\right)$ periodisk med perioden 4π



$f(x) =$ jämn funktion $\Rightarrow \boxed{b_k = 0}$

Fourierserien: av $f(x)$

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,1\,.dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right)dx \qquad \frac{x}{2} = y \quad \frac{1}{2}dx = dy \quad dx = 2dy$$

$$\underbrace{-\pi < x < \pi}\ \Leftrightarrow\ -\frac{\pi}{2} < \overbrace{\frac{x}{2}}^{y} < \frac{\pi}{2}$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(y)\,2dy =$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos y\,dy = \frac{2}{\pi}\Big[\sin y\Big]_{y=-\pi/2}^{y=\pi/2} = \frac{2}{\pi}\Big[\sin\left(\frac{\pi}{2}\right) \underbrace{-\sin\left(-\frac{\pi}{2}\right)}_{+}\Big]$$

$$= \frac{2}{\pi}\cdot 2 = \frac{4}{\pi} \qquad \boxed{a_0 = \frac{4}{\pi}}$$

$$\boxed{a_0 = \frac{\ }{\pi}}$$

$$a_k = \langle f(x), \cos kx \rangle, \quad k = 1, 2, \ldots$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\cos\left(\frac{x}{2}\right)}^{f(x)} \cos(kx)\, dx$$

$$\boxed{\cos(\alpha)\cdot\cos\beta = \cos\left(\frac{x}{2}\right)\cos(kx)}$$

$$\begin{cases} \cdot \ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cdot \ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{cases}$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

Då $\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$$\begin{cases} \frac{x}{2} + kx = x\left(k+\frac{1}{2}\right) \\ \frac{x}{2} - kx = x\left(\frac{1}{2}-k\right) = -x\left(k-\frac{1}{2}\right) \end{cases} \Leftarrow$$

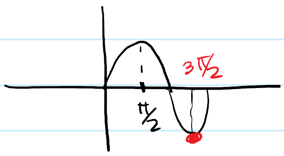
$$\boxed{\cos\left(\frac{x}{2}\right)\cdot\cos(kx) = \frac{1}{2}\cos\left(\left(k+\frac{1}{2}\right)x\right) + \frac{1}{2}\cos\left(\left(k-\frac{1}{2}\right)x\right)} \ (*)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}\right)\cos(kx)\,dx \overset{(*)}{=} \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\left(\left(k+\frac{1}{2}\right)x\right)dx}_{(A)} + \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\left(\left(k-\frac{1}{2}\right)x\right)dx}_{(B)}$$

(A) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(\left(k+\frac{1}{2}\right)x\right)dx = \frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{\pi}\cos\left(\left(k+\frac{1}{2}\right)x\right)dx$

$$\frac{1}{\pi}\left[\frac{\sin\left(\left(k+\frac{1}{2}\right)x\right)}{\left(k+\frac{1}{2}\right)}\right]_{x=0}^{x=\pi} = \boxed{\frac{1}{\pi}(-1)^k}$$

$$\underbrace{\sin\left(k\pi + \frac{\pi}{2}\right)} = (-1)^k$$

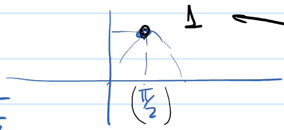


$$\frac{\pi}{2} + 1\pi = \frac{3\pi}{2}$$

$$\underbrace{\sin\left(\frac{3\pi}{2}\right) = -1 = (-1)^1}$$

B) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(\left(k-\frac{1}{2}\right)x\right)dx = \frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{\pi}\underbrace{\cos\left(\left(k-\frac{1}{2}\right)x\right)}dx$

$$\frac{1}{\pi}\left[\frac{\sin\left(\left(k-\frac{1}{2}\right)x\right)}{\left(k-\frac{1}{2}\right)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin\left(\pi - \frac{1}{2}\pi\right)}{k-\frac{1}{2}}\right] = \frac{1}{\pi}\boxed{(-1)^{k-1}}$$



$$1\pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \quad (k=1) \Rightarrow (-1)^{k-1} = (-1)^0 = 1$$

$$(-1)^1 = -1$$

Då $a_k = \frac{1}{\pi}\frac{(-1)^k}{(k+\frac{1}{2})} + \frac{1}{\pi}\frac{(-1)^{k-1}}{(k-\frac{1}{2})}$

Då $a_k = \frac{1}{\pi}\frac{(-1)^k}{(k+\frac{1}{2})} + \frac{1}{\pi}\frac{(-1)^{k-1}}{(k-\frac{1}{2})}$

$(-1)^1 = -1$

$\boxed{k-1}$

$k=1 \Rightarrow k-1=0$
$(-1)^0 = 1$

$$a_k = \frac{1}{\pi}\left[\frac{(-1)^{k-1}\cdot(-1)}{k+\frac{1}{2}} + \frac{(-1)}{k-\frac{1}{2}}\right] =$$

$$a_k = \frac{(-1)^{k-1}}{\pi}\left[\frac{-1}{k+\frac{1}{2}} + \frac{1}{k-\frac{1}{2}}\right] =$$

$$a_k = \frac{(-1)^{k-1}}{\pi}\left[\frac{-(k-\frac{1}{2}) + (k+\frac{1}{2})}{(k+\frac{1}{2})(k-\frac{1}{2})}\right] =$$

$$a_k = \frac{(-1)^{k-1}}{\pi}\left[\frac{\frac{1}{2}+\frac{1}{2}}{(k^2-\frac{1}{4})}\right] \qquad k = 1, 2, \ldots$$

$$a_k = \frac{(-1)^{k-1}}{\pi}\left[\frac{1}{k^2-\frac{1}{4}}\right]$$

Då $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$

$$f(x) \sim \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi\,(k^2-\frac{1}{4})} \cos(kx)$$

$$f(x) \sim \underbrace{\frac{2}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k^2-\frac{1}{4})}\underbrace{\cos(kx)}}_{S(x)}$$

2007 b : $S(x)$ summan av Fourierserie

$$f(x) \sim \frac{2}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\underbrace{(k^2-\frac{1}{4})}}\cos(kx) \qquad x \in [-\pi, \pi]$$

$f(x) = \cos(\frac{x}{2}) \quad \forall x \in \mathbb{R}$

$$\Rightarrow \quad S(x) = f(x) = \cos\left(\frac{x}{2}\right) \quad x \in [-\pi, \pi]$$

Periodisk förtsättning:

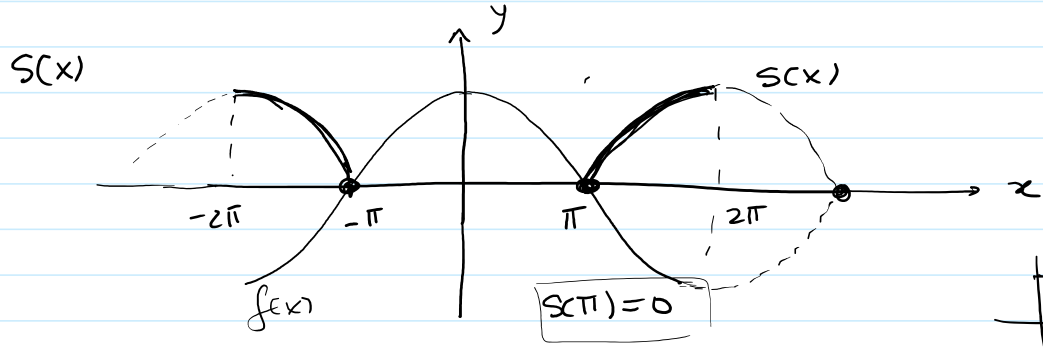
$$f(x+2\pi) = \cos\left(\frac{x+2\pi}{2}\right) = \cos\left(\frac{x}{2}+\pi\right) \overset{*}{=} -\cos\left(\frac{x}{2}\right)$$

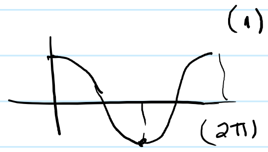
$$\underbrace{S(x) = S(x+2\pi)} \quad 2\pi\text{-periodisk}$$

där for att alla bas funktioner är 2π-periodisk.

$$\cos\left(\frac{x}{2}\right) = S(x) = S(x+2\pi) = f(x+2\pi) = -\cos\left(\frac{x}{2}\right)$$

$$\boxed{S(x) = \left|\cos\left(\frac{x}{2}\right)\right|}$$





$\cos(\pi) = (-1)$

2007C) Beräkna $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k^2-\frac{1}{4})}$ $\quad \left(\sum \frac{1}{k^2}\right) \quad (-1)^k$

$$S(x) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k^2-\frac{1}{4})} \cos(xk)$$

$x = \pi$

$$S(\pi) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\cdot\cos(\pi\cdot k)}{(k^2-1/4)}$$

$$\frac{2}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\cdot(-1)^{k}}{(k^2-\frac{1}{4})}$$

$$\pi \qquad \pi \quad k=1 \qquad \overline{(k^2-\frac{1}{4})}$$

$$(-1)^{k-1}\cdot(-1)^{k-1}\cdot(-1)$$

$$\underbrace{(+)\quad(-1)}_{-1}$$

$$S(\pi) = \frac{2}{\pi} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{(k^2-\frac{1}{4})}$$

$$\| $$

$$0$$

$$+\frac{1}{\cancel{\pi}}\sum_{k=1}^{\infty}\frac{1}{(k^2-1/4)} = \frac{2}{\cancel{\pi}}$$

$$\boxed{\underbrace{\sum_{k=1}^{\infty}\frac{1}{(k^2-\frac{1}{4})}}_{} = 2}$$

Konvergent enligt jämförelsekriteriet