V8_20maj2021

den 20 maj 2021 11:37

Fourier serie

Uppgift 2008 sidan 132 - Del 3 ELW

a) $f(x) = x^2 \quad x \in [0, 2\pi)$ perioden 2π

Bestäm fourierserie till $f(x)$

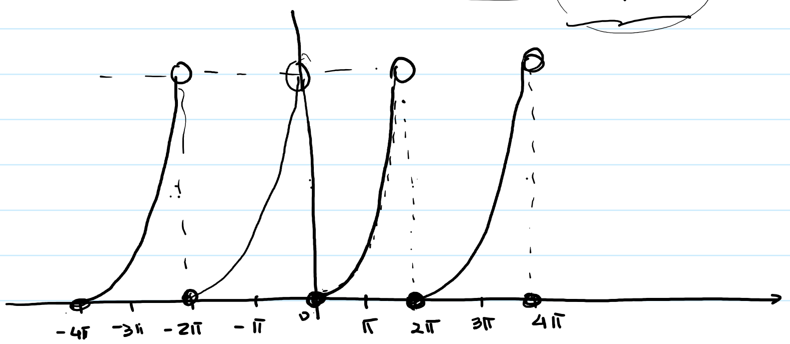
b) Använd fourierserie för att beräkna $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$f(x) = x^2 \ , \ [0, 2\pi)$

$(\pi^2) \simeq 9..$

$(2\pi)^2 \doteq 4 \cdot 9 \gtrsim 36 ..$

Lösning:



$x = 0$ f är inte kontinuerlig

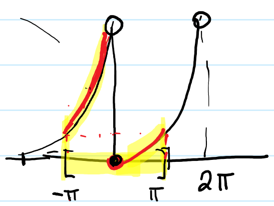
$x = 2\pi \cdot k, \quad k \in \mathbb{Z}$ då är f inte kontinuerlig.

$$S(x) = \lim_{n\to\infty} S_n \ , \quad S_n = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos k + b_k \sin kx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

$$S(x) = \begin{cases} f(x) & x \in \underbrace{(0, 2\pi), \ (-2\pi, 0), \ \dots}_{(k2\pi, (k+1)2\pi)} \ , \ k \in \mathbb{Z}. \\ \frac{1}{2}\left(f(x^+) + f(x^-)\right) & x = 0, \ 2\pi, \ -2\pi, \ \dots \end{cases}$$

$\boxed{x = k \cdot 2\pi \ , \quad k \in \mathbb{Z}.}$



$$f(x) = \begin{cases} x^2 \ , & x \in [0, \pi) \\ (x+2\pi)^2 \ , & x \in [-\pi, 0) \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2\,dx + \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(x+2\pi)^3}{3}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - \frac{(-\pi+2\pi)^3}{3}\right] + \frac{1}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{8\pi^3}{3} - \cancel{\frac{\pi^3}{3}}\right] + \cancel{\frac{1}{\pi}\frac{\pi^3}{3}} = \frac{8\pi^2}{3}$$

$$\boxed{\frac{a_0}{2} = \frac{4\pi^2}{3}}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \underbrace{\frac{1}{\pi}\int_{-\pi}^{0}(x+2\pi)^2\cos kx\,dx}_{A} + \underbrace{\frac{1}{\pi}\int_{0}^{\pi}x^2\cos kx\,dx}_{B}$$

(B) $\frac{1}{\pi}\int_0^{\pi} x^2 \cos kx\,dx$ $\quad \begin{cases} u = x^2 \to du = 2x \\ dv = \cos kx \to v = \frac{\sin kx}{k}\end{cases}$ $\quad \left[u.v\Big|_0^{\pi} - \int_0^{\pi} du\,v\right]_0^{\pi}$

$$= \cancel{\frac{1}{\pi}\left[x^2\frac{\sin kx}{k}\right]_0^{\pi}} - \frac{1}{\pi}\int_0^{\pi} 2x\,\frac{\sin kx}{k}\,dx$$

$$-\frac{2}{k\pi}\int_0^{\pi} x\sin kx\,dx = \begin{cases} u = x & du = 1 \\ dv = \sin kx & v = -\frac{\cos kx}{k}\end{cases}$$

$$\underbrace{\cancel{-}\frac{2}{k\pi}\left[\cancel{-}x\frac{\cos kx}{k}\right]_{x=0}^{\pi}}_{=\left[\frac{2\pi}{k^2}\cos k\pi - 0\right]} \cancel{-} \underbrace{\frac{2}{\pi k^2}\int_0^{\pi}\cancel{-}\cos kx\,dx}_{\frac{2}{\pi k^2}\cancel{\left[\frac{\sin kx}{k}\right]_0^{\pi}}}$$

$$\frac{2\pi}{k^2}(-1)^k$$

$a_k = A + B$,

$$\boxed{B = \frac{2}{k^2}(-1)^k}$$

$$a_k = A + B \qquad \boxed{B = \frac{2}{k^2}(-1)^k}$$

(A) $$\frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2 \cos kx \, dx$$

$$\left\{\begin{array}{ll} u = (x+2\pi)^2 & du = 2(x+2\pi) \\ dv = \cos kx & v = \frac{\sin kx}{k} \end{array}\right.$$

$$= \frac{1}{\pi}\left[\cancel{(x+2\pi)^2 \frac{\sin k\pi}{k}}\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0} 2(x+2\pi)\frac{\sin kx}{k}\,dx$$

$$= 0 - \frac{2}{k\pi}\int_{-\pi}^{0} (x+2x)\sin kx\,dx$$

$$\left\{\begin{array}{ll} u = (x+2\pi) & du = 1 \\ dv = \sin kx & v = -\frac{\cos kx}{k} \end{array}\right.$$

$$\cancel{-}\frac{2}{k\pi}\left[\cancel{-}(x+2\pi)\frac{\cos kx}{k}\right]_{-\pi}^{0} + \cancel{\frac{2}{k\pi}\int_{-\pi}^{0} -\frac{\cos kx}{k}\,dx}$$

$$= \frac{2}{k\pi}\left[\frac{2\pi}{k}\cdot 1 - (\underbrace{-\pi+2\pi}_{\pi})\frac{\cos(-\pi k)}{k}\right] =$$

$$= \frac{2}{k\cancel{\pi}}\left[\frac{2\cancel{\pi}}{k} - \frac{\cancel{\pi}(-1)^k}{k}\right] = \frac{4}{k^2} - \frac{2}{k^2}(-1)^k$$

$$a_k = \underbrace{\left(\frac{4}{k^2} - \cancel{\frac{2}{k^2}(-1)^k}\right)}_{A} + \underbrace{\left(\cancel{\frac{2}{k^2}(-1)^k}\right)}_{B} = \boxed{\frac{4}{k^2}} = \boxed{a_k}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \underbrace{\frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2 \sin kx\,dx}_{A} + \underbrace{\frac{1}{\pi}\int_{0}^{\pi} x^2 \sin kx\,dx}_{B}$$

(A) $$\frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2 \sin kx\,dx$$

$$\left\{\begin{array}{ll} u = (x+2\pi)^2 \to & du = 2(x+2\pi) \\ dv = \sin kx & v = -\frac{\cos kx}{k} \end{array}\right.$$

$$\frac{1}{\pi}\left[(x+2\pi)^2\left(-\frac{\cos kx}{k}\right)\right]_{-\pi}^{0} + \frac{1}{k\pi}\int_{-\pi}^{0} 2(x+2\pi)\cos kx\,dx =$$

$$\left\{\begin{array}{l} u = (x+2\pi) \\ du = 1 \\ dv = \cos kx \\ v = \frac{\sin kx}{k} \end{array}\right.$$

$$\frac{1}{\cancel{\pi}}\left[-\frac{4\pi^{\cancel{2}}}{k} + (-\pi+2\pi)\frac{\cos(-\pi k)}{k}\right] + \frac{2}{k\pi}\left[\cancel{(x+2\pi)\frac{\sin k\pi}{k}}\right]_{-\pi}^{0}$$

$$| \qquad\qquad\qquad\qquad 0$$

$$- \frac{2}{k^2\pi}\int_{-\pi}^{0} \sin kx \, dx$$

Ⓐ: $\left[-\frac{4\pi}{k} + \frac{\pi}{k}(-1)^k \right] \quad \cancel{-} \frac{2}{k^2\pi}\left[\cancel{-}\frac{\cos kx}{k} \right]_{-\pi}^{0}$

$$+ \frac{2}{\pi k^3}\left[1 - \cos(k(-\pi)) \right] =$$

$$\frac{2}{\pi \cdot k^3}\left[1 - (-1)^k \right]$$

Ⓐ $\left[-\frac{4\pi}{k} + \frac{\pi}{k}(-1)^k \right] + \frac{2}{\pi k^3}\left[1 - (-1)^k \right]$

Ⓑ $\frac{1}{\pi}\int_0^{\pi} x^2 \sin kx \, d =$

$-\frac{1}{k\cancel{\pi}}\left[\pi^{\cancel{2}} \cos k\pi - 0 \right] + \frac{2}{k\pi}\int_0^{\pi} x \cos kx \, dx \overset{\circledast}{=}$ $\begin{cases} u = x \to du = 1 \\ dv = \cos kx \quad v = \frac{\sin kx}{k} \end{cases}$

$\overset{\circledast}{=} -\frac{1}{k}(-1)^k \cdot \pi + \frac{2}{k\pi}\left[\cancel{\left[\frac{x \sin kx}{k} \right]_0^{\pi}} - \int_0^{\pi} 1 \cdot \frac{\sin xk}{k} \, dx \right] =$

$= -\frac{(-1)^k \pi}{k} - \frac{2}{k^2\pi}\left[\int_0^{1} \sin kx \, dx \right] =$

$= -(-1)^k \frac{\pi}{k} - \frac{2}{k^2\pi}\left[-\frac{\cos kx}{k} \right]_0^{\pi} =$

$= -(-1)^k \frac{\pi}{k} + \frac{2}{k^3\pi}\left[(-1)^k - 1 \right]$

Då

$$b_k = \frac{1}{k}\left[-4\pi + \cancel{\pi(-1)^k} \right] + \frac{2}{\pi k^3}\underbrace{\left[\cancel{1 - (-1)^k} \right] + \cancel{-\frac{\pi(-1)^k}{k}} + \frac{2}{k^2\pi}\left[\cancel{(-1)^k - 1} \right]}$$

$$\boxed{b_k = -\frac{4\pi}{k}}$$

Och fourierserien till $f(x)$ är:

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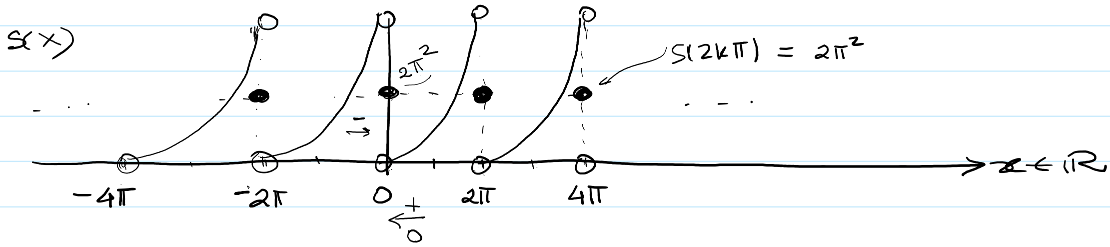
$$f(x) \sim S(x) = \frac{4\pi^2}{3} + \sum_{k=1}^{\infty} \left(\frac{4}{k^2}\right)\cos kx + \left(-\frac{4\pi}{k}\right)\sin kx$$

$$\boxed{f(x) \sim S(x) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}\cos kx - \frac{\pi}{k}\sin kx}$$

$S(x) =$ summan

$$S(x) = \begin{cases} f(x), & \forall x \in (k(2\pi), (k+1)2\pi) \quad k \in \mathbb{Z} \\ \frac{1}{2}\left(f(x^+) + f(x^-)\right), & x = k(2\pi), \quad k \in \mathbb{Z}. \end{cases}$$



$$\boxed{S(0) \overset{*}{=} \frac{1}{2}\left(f(0^+) + f(0^-)\right) = \frac{1}{2}\left(0 + 4\pi^2\right) = 2\pi^2}$$

b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ med hjälp av fourierserie till $f(x)$ (a)

Lösning.

$$S(x) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}\left(\cos(kx) - \pi k \sin kx\right)$$

$\underline{x = 0}$

$$S(0) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}\left(\underbrace{\cos(0)}_{1} - \cancel{\pi \cdot 0 \cdot \sin 0}\right)$$

$$S(0) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$\|$

$$S(0) \overset{*}{=} 2\pi^2$$

$S(0) = \text{''}$

$$\Rightarrow \quad \boxed{2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}}$$

$p > 1$.

Konvergent serien.

Då kan man göra operationer med konvergenta serier. :

$$6\pi^2 - 4\pi^2 = 12 \sum_{k=1}^{\infty} \frac{1}{k^2} \Longrightarrow$$

$$\frac{\cancel{2}\pi^2}{\cancel{12}} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6}$$