Föreläsning 5

Trigonometriska intergaler

Trigonometriska integraler

$\sin^2 x + \cos^2 x = 1$

Exempel 1:

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = \int \underbrace{(1-\cos^2 x)}_{1-u^2} \underbrace{\sin x\,dx}_{-du} = \left| \begin{matrix} u = \cos x \\ du = -\sin x\,dx \end{matrix} \right| =$$

$$= -\int (1-u^2)\,du = -\int du + \int u^2\,du = -u + \frac{u^3}{3} = -\cos x + \frac{\cos^3 x}{3} + C$$

$a^5 = a^4 \cdot a = (a^2)^2 a$

Exempel 2:

$$\int \sin^5 x \cos^2 x\,dx = \int (1-\cos^2 x)^2 \cdot \sin x \cdot \cos^2 x\,dx = \left| \begin{matrix} u = \cos x \\ du = -\sin x\,dx \end{matrix} \right| =$$

$\sin^5 x = (\sin^2 x)^2 \cdot \sin x$

$= (1-\cos^2 x)^2 \sin x$

$$= -\int (1-u^2)^2 u^2\,du = -\int (1-2u^2+u^4) u^2\,du =$$

$$= -\int (u^2 - 2u^4 + u^6)\,du = -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) = -\left(\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7}\right) + C$$

Exempel 3:

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx = \frac{x}{2} -$$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\left| \begin{matrix} 2x = t \\ dx = dt/2 \end{matrix} \right|$

$$-\frac{1}{2}\int \cos t \cdot \frac{dt}{2} = \frac{x}{2} - \frac{1}{4}\sin t = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

Exempel 4: $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx =$

$\sin 2x = 2 \sin x \cos x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

$\cos 2x = \cos^2 x - \sin^2 x$

(1) $\sin^2 x \cos^2 x = (\sin x \cos x)^2 = \left(\dfrac{\sin 2x}{2}\right)^2 = \dfrac{(\sin 2x)^2}{4} = \dfrac{1}{4} \cdot \dfrac{1-\cos 4x}{2}$

$$\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx = \int_0^{\pi/2} \frac{1-\cos 4x}{8} \, dx = \frac{1}{8}\left(\int_0^{\pi/2} dx - \int_0^{\pi/2} \cos 4x \, dx \right) =$$

$$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x \right)\Big|_0^{\pi/2} = \frac{\pi}{16} \qquad \left(\sin 4 \cdot \frac{\pi}{2} = \sin 2\pi = 0 \right)$$

(2) $\sin^2 x \cos^2 x = \dfrac{1-\cos 2x}{2} \cdot \dfrac{1+\cos 2x}{2} = \dfrac{1-(\cos 2x)^2}{2}$ (?)

<u>Exempel 5:</u> $\int \sin 6x \cos 8x \, dx = \int \frac{1}{2}(\sin(-2x) + \sin(14x))\,dx = -\frac{1}{2}\int \sin 2x\,dx + \frac{1}{2}\int \sin 14x\,dx$

$= +\frac{1}{2}\cdot\frac{1}{2}\cos 2x - \frac{1}{2}\cdot\frac{1}{14}\cos 14x =$

$= \frac{1}{4}\cos 2x - \frac{1}{28}\cos 14x + C$

$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$\sin A \sin B =$

$\cos A \cos B =$

$$\int \sin 2x\,dx = \left| \begin{matrix} t = 2x \\ dt = 2dx, \; dx = \frac{dt}{2} \end{matrix} \right| = \int \sin t \cdot \frac{dt}{2} =$$

$$= \frac{1}{2}\int \sin t\,dt = -\frac{1}{2}\cos t = -\frac{1}{2}\cos 2x + C$$

$$\int \sin 14x\,dx = \left| \begin{matrix} t = 14x \\ \dots \end{matrix} \right| = -\frac{1}{14}\cos 14x + C$$

Exempel 6 (sidan 485, uppgift 55): $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$ genomsnittlig värde av funktion f på intervallet $[a,b]$

$$\bar{f} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x \, dx$$

(exempel 2)

$\cos^2 x = 1 - \sin^2 x$

Trigonometrisk substitution

$$a^2 - a^2\sin^2 t = a^2(1-\sin^2 t) = a^2\cos^2 t$$

$$\sqrt{a^2\cos^2 t} = |a\cos t| = a\cos t$$

Exempel 7: $\int_0^a \sqrt{a^2-x^2}\,dx = \left| \begin{array}{l} x = a\sin t, \; dx = a\cos t\,dt \\ x=0, \; a\sin t = 0 \Rightarrow t=0 \\ x=a, \; \cancel{a}\sin t = \cancel{a}_1 \Rightarrow t = \frac{\pi}{2} \end{array} \right| = \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2 t}\; a\cos t\,dt$

$$= \int_0^{\pi/2} a\cos t \cdot a\cos t\,dt = a^2 \int_0^{\pi/2} \cos^2 t\,dt = a^2\int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{a^2}{2}\int_0^{\pi/2} dt + \frac{a^2}{2}\int_0^{\pi/2} \cos 2t\,dt = \frac{a^2\pi}{4} + \frac{a^2}{4}\sin 2t \Big|_0^{\pi/2} = \frac{a^2\pi}{4}$$

Exempel 8:

① $$\int_0^a x\sqrt{a^2-x^2}\,dx = \left| \begin{matrix} t=a^2-x^2, & dt=-2x\,dx \\ x=0, t=a^2; & x=a, t=0 \end{matrix} \right| = -\frac{1}{2}\int_{a^2}^{0}\sqrt{t}\,dt = -\frac{1}{\cancel{2}}\cdot\frac{t^{3/2}}{3/\cancel{2}}\Big|_{a^2}^{0} = -\frac{1}{3}t^{3/2}\Big|_{a^2}^{0}$$

$$= -\frac{1}{3}\left(0-(a^{\cancel{2}})^{3/\cancel{2}}\right) = \frac{a^3}{3}$$

② $$\int_0^a x\sqrt{a^2-x^2}\,dx = \left| \begin{matrix} x=a\sin t, \; dx=a\cos t\,dt \\ x=0, \; a\sin t=0 \Rightarrow t=0 \\ x=a, \; \cancel{a}\sin t=\cancel{a}_1 \Rightarrow t=\frac{\pi}{2} \end{matrix} \right| = \int_0^{\pi/2} a\sin t\cdot a\cos t\cdot a\cos t\,dt =$$

$$= a^3\int_0^{\pi/2}\sin t\cdot\cos^2 t\,dt = \left| \begin{matrix} u=\cos t, & du=-\sin t\,dt \\ t=0, u=1; & t=\frac{\pi}{2}, u=0 \end{matrix} \right| =$$

$$= -a^3\int_1^0 u^2\,du = -a^3\frac{u^3}{3}\Big|_1^0 = \frac{a^3}{3}$$

Exempel 9: $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \left| \begin{array}{l} x=\sin t\,, \quad -\pi/2 \le t \le \pi/2 \\ dx = \cos t\,dt \end{array} \right| =$

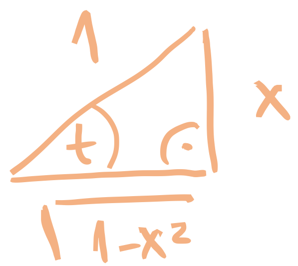
$|\cos t| = \cos t$

$$= \int \frac{\sqrt{1-\cos^2 t}}{\sin^2 t}\cos t\,dt = \int \frac{|\cos t|}{\sin^2 t}\cos t\,dt = \int \frac{\cos^2 t}{\sin^2 t}\,dt =$$

$$= \int \frac{1-\sin^2 t}{\sin^2 t}\,dt = \int \frac{dt}{\sin^2 t} - \int dt = -\cot t - t =$$

$$\frac{d}{dx}(\cot t) = \frac{d}{dx}\left(\frac{\cos t}{\sin t}\right) = \frac{-\sin^2 t - \cos^2 t}{\sin^2 t} = -\frac{1}{\sin^2 t}$$

$$= -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C$$



$$\cot t = \frac{\cos t}{\sin t} = \frac{\sqrt{1-x^2}}{x}$$