# Föreläsning 10

Separabla differentialekvationer

## Exempel 1:

$$\frac{du}{dt} = \frac{1+t^4}{\underbrace{ut^2+u^4t^2}_{(u+u^4)t^2}}$$

$$u^2t^3 + tu$$

$$(u+u^4)\,du = \frac{1+t^4}{t^2}\,dt$$

$$\frac{u^2}{2} + \frac{u^5}{5} = \int \frac{dt}{t^2} + \int t^2 dt = -\frac{1}{t} + \frac{t^3}{3} + C$$

## Exempel 2:

$$\frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2$$

$$y\,dy = \frac{\ln x}{x}\,dx$$

$$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$$

$$y^2 = (\ln x)^2 + C$$

$$y = \pm\sqrt{(\ln x)^2 + C}$$

$$2 = \sqrt{\underbrace{(\ln 1)^2}_{= 0} + C}$$

$$\Downarrow$$

$$C = 4$$

$$\Downarrow$$

$$y = \sqrt{(\ln x)^2 + 4}$$

Exempel 3:

$$y' \tan x = a+y \quad , \quad y\left(\frac{\pi}{3}\right) = a \, , \quad 0 < x < \frac{\pi}{2}$$

$$\frac{dy}{a+y} = \frac{\cos x}{\sin x}\, dx$$

$\sin x > 0$

$$\ln|a+y| = \ln|\sin x| + C$$

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$$e^{\ln|a+y|} = e^{\ln \sin x + C}$$

$$|a+y| = C \sin x$$

$$a+y = C \sin x$$

$$y\left(\frac{\pi}{3}\right) = a$$

$$a+a = C \sin \frac{\pi}{3}$$

$$2a = C \cdot \frac{\sqrt{3}}{2}$$

$$\boxed{C = \frac{4a}{\sqrt{3}}}$$

$$y = \frac{4a}{\sqrt{3}} \sin x - a$$

## Exempel 4:

$y' = 2x\sqrt{1-y^2}$

a) hitta lösning

b) $y(0)=0$

c) $y(0)=2$

$y=\pm 1 \Rightarrow y'=0$

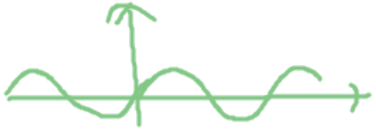
$0 = 2x\sqrt{1-1} \to$ OK

$y=\pm 1$ lösning också

a) $\dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx$

$\arcsin y = x^2 + C$

Kan vi skriva $y = \sin(x^2+C)$?



$\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

$\arcsin x : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Nej, vi vet inte om $x^2+C \in [-1,1]$.

b) $y(0)=0$

om $x \in [-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}]$

$\arcsin 0 = 0^2 + C \Rightarrow C=0$

$y = \sin(x^2)$

c) $y(0)=2$

$2 = \sin(0^2+C)$

har ingen lösning

**Exempel 5:** Hitta $f(x)$ så att:

$$f'(x) = xf(x) - x \quad , \quad f(0) = 2$$

$$y = f(x) \; , \; y' = f'(x)$$

$$y' = x(y-1) \quad , \quad y(0) = 2$$

$$\frac{dy}{y-1} = x\,dx$$

$$\ln|y-1| = \frac{x^2}{2} + C$$

$$e^{\ln|y-1|} = e^{\frac{x^2}{2} + C}$$

$$y - 1 = Ce^{x^2/2}$$

$$y = Ce^{x^2/2} + 1$$

$$2 = Ce^0 + 1$$

$$2 = C + 1$$

$$C = 1$$

$$\Downarrow$$

$$y = e^{x^2/2} + 1$$

$y = 1 \Rightarrow y' = 0 \Rightarrow 0 = x(1-1)$ ok

$\Rightarrow y = 1$ är lösning till allmän problem (ODE utan villkor)

$y(0) = 2 \Rightarrow y(x) = 1$ är inte lösning till problem

$$\begin{cases} y' = x(y-1) \\ y(0) = 2 \end{cases}$$

## Exempel 6:

$$y(x) = 2 + \int_2^x (t - t\,y(t))\,dt$$

1. derivera $\rightarrow$ ODE
2. lösa ODE
3. hitta $C$

$$y' = \frac{d}{dx}\left(\int_2^x (t - t\,y(t))\,dt\right) = x - xy$$

$$y' = x(1-y)$$

$$\frac{dy}{1-y} = x\,dx \quad , \quad y \neq 1$$

$$-\ln|1-y| = \frac{x^2}{2} + C$$

$$1 - y = Ce^{-x^2/2}$$

$$y = 1 + Ce^{-x^2/2}$$

$$y = 1 , \quad 1 = 2 + \int_2^x (t - t \cdot 1)\,dt = 2$$

omöjligt $\Rightarrow y \neq 1$

$x = 2$

$$y(2) = 2 + \underbrace{\int_2^2 (t - t\,y(t))\,dt}_{=0} \Rightarrow y(2) = 2$$

$$2 = 1 + Ce^{-2} \Rightarrow C = \ldots$$

$$\Rightarrow y = \ldots$$

## Exempel

$f(3)=2$ , $(t^2+1)f'(t) + (f(t))^2 + 1 = 0$ , $t \neq 1$

$y = f(t)$ , $y' = f'(t)$

$$(t^2+1)y' + y^2 + 1 = 0$$

$$(t^2+1)y' = -y^2 - 1$$

$$\frac{dy}{-y^2-1} = \frac{dt}{t^2+1}$$

$$\frac{dy}{1+y^2} = -\frac{dt}{1+t^2}$$

$$\arctan y = -\arctan t + C$$

$$\arctan y + \arctan t = C$$

förenkl.

$$\arctan \frac{y+t}{1-yt} = C$$

$$\tan\left(\arctan \frac{y+t}{1-yt}\right) = \tan C = C$$

$$\boxed{e^C = C \,,\ \tan C = C}$$

$f(3)=2$

$$\arctan \frac{2+3}{1-2\cdot 3} = \arctan(-1) = -\pi/4$$

$$\arctan \frac{y+t}{1-yt} = -\frac{\pi}{4}$$

$$\tan\left(\arctan \frac{y+t}{1-yt}\right) = \tan\left(-\frac{\pi}{4}\right)$$

$$\frac{y+t}{1-yt} = -1 \Rightarrow y = \dots$$