Förläsning 11

Linjära ODE

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$$\frac{dy}{dx} + P(x)y = Q(x)$$

linjär första ordningen

P, Q kontinuerliga funktioner

$$y' + \frac{x}{y^2} = 5$$ icke-linjär ODE

$$y'' + y' + y = x$$ linjär, andra ordning

Exempel 1: $xy' + y = 2x, \quad x \neq 0$

$$y' + \frac{1}{x} y = 2$$

$$(xy)' = xy' + y = 2x$$

$$\int (xy)' dx = \int 2x dx$$

$$xy = x^2 + C$$

$$y = x + \frac{C}{x}$$ → lösning

$y' + P(x)y = Q(x)$ / multiplicera med någon $I(x)$

så att vi har $(I(x)y)'$

$I(x)$ integrerande faktor $(\mu(x))$

$(xy)' = xy' + x$

① $I(x)(y' + P(x)y) = I(x)Q(x)$

$$\underbrace{I(x)(y'+P(x)y)}_{=(I(x)y)'}$$

$$\int (I(x)y)'dx = \int I(x)Q(x)dx$$

$$I(x)y = \int I(x)Q(x)dx + C$$

$$y = \frac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right)$$

lösning

② för att hitta $I(x)$:

$$I(x)(y' + P(x)y) = \cancel{I(x)y'} + I(x)P(x)y$$

$$\| $$

$$(I(x)y)' = I'(x)y + \cancel{I(x)y'}$$

$$I'(x)y = I(x)P(x)y \quad , y \neq 0$$

$$I'(x) = I(x)P(x) \quad \text{separabel ODE}$$

$$\frac{dI}{I} = P(x)dx$$

$$\ln|I| = \int P(x)dx \Rightarrow$$

$$I(x) = Ae^{\int P(x)dx} \quad , A = 1$$

Exempel 2:

exp. $= 1 \Rightarrow$ linjär

$$\cancel{y' + 3x^2 y^{2}} = 6x^2$$

$$\frac{dy}{dx} + 3x^2 y = 6x^2$$

$$P(x) = 3x^2 \,,\quad Q(x) = 6x^2$$

$$\text{IF:}\quad I(x) = e^{\int P(x)dx}$$

!

$$\int 3x^2 dx = x^3 \,,\quad I(x) = e^{x^3}$$

$$\underbrace{e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y}_{=\frac{d}{dx}\left(e^{x^3} y\right)} = 6x^2 e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3} y\right) = 6x^2 e^{x^3}$$

$$\int \frac{d}{dx}\left(e^{x^3} y\right) dx = \int 6x^2 e^{x^3} dx = \left| \begin{matrix} u = x^3 \\ du = 3x^2 dx \end{matrix} \right| = 2e^{x^3}$$

$$e^{x^3} y = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

Exempel 3:

$P(x) \to I(x) = e^{\int P(x)\,dx}$

$x^2 y' + xy = 1, \quad x > 0, \quad y(1) = 2$

$y' + \frac{1}{x} y = \frac{1}{x^2}$

IF: $I(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = e^{\ln x} = x$

$xy' + y = \frac{1}{x}$

$(xy)' = \frac{1}{x}$

$\int (xy)' dx = \int \frac{dx}{x}$

$xy = \ln x + C$

$y = \frac{\ln x}{x} + \frac{C}{x}$

$2 = \frac{\ln 1}{1} + \frac{C}{1} \Rightarrow C = 2$

$\ln|y|$

$y = \frac{\ln x + 2}{x}$

om $I(x)$ är I.F. då

$(I(x)y)' = I(x)Q(x)$

1. hitta $P(x)$
2. beräkna $I(x)$
3. $(I(x)y)' = I(x)Q(x)$
4. integrera
5. bestämma C

Exempel 4:

$xy' = y + x^2\sin x$, $y(\pi) = 0$

$xy' - y = x^2\sin x$

$y' - \frac{1}{x}y = x\sin x$, $x \neq 0$

IF: $I(x) = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = e^{\ln x^{-1}} = \frac{1}{x}$ om $x>0$

$x<0$,

$\ln|x| = \ln(-x)$, $e^{-\ln(-x)} = e^{\ln(-x)^{-1}} = -\frac{1}{x}$

$\ln|x| \to$ $x>0$, $\cancel{x<0}$

$\left(\frac{1}{x}y\right)' = \frac{1}{x}\cdot x\sin x = \sin x$ / integrera

$\frac{1}{x}y = -\cos x + C$

$y = -x\cos x + Cx$

$y(\pi) = 0$

$0 = -\pi\cdot\underbrace{\cos\pi}_{-1} + C\pi$

$0 = \pi + C\pi$

$C = -1$

$\Rightarrow y = -x\cos x - x$, $x>0$

om $I(x)$ är I.F.

$\Rightarrow (I(x)y)' = \text{höger sidan} \cdot I(x)$

Exempel

$$y' + 2xy = 1$$

IF: $I(x) = e^{\int 2x\,dx} = e^{x^2}$

$$(e^{x^2}y)' = e^{x^2}$$

$$\int (e^{x^2}y)\,dx = \int e^{x^2}\,dx$$

$$e^{x^2}y = \int e^{x^2}\,dx + C$$

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

$$y = e^{-x^2}\left(\int e^{x^2}\,dx + C\right)$$