Lösningsförslag övningstenta maj 2021

1. a) $$\int \frac{9}{x^2-12x}\,dx = \int \frac{9}{x(x-12)}\,dx = -\frac{3}{4}\int\frac{dx}{x} + \frac{3}{4}\int\frac{dx}{x-12} = -\frac{3}{4}\cdot \ln|x| + \frac{3}{4}\ln|x-12| =$$
$$= \frac{3}{4}\ln\left|\frac{x-12}{x}\right| = \frac{3}{4}\ln\left|1-\frac{12}{x}\right| + C$$

$$\left(\frac{9}{x^2-12x} = \frac{9}{x(x-12)} = \frac{A}{x} + \frac{B}{x-12} = \frac{A(x-12)+Bx}{x(x-12)} \Rightarrow \left.\begin{aligned} 9 &= -12A \\ 0 &= A+B \end{aligned}\right\} \Rightarrow A = -\frac{3}{4},\ B = \frac{3}{4}\right)$$

b) $$\int \sqrt{1-7t^2}\,dt = \left| \begin{array}{l} t = \frac{1}{\sqrt{7}}\sin x \ , \ dt = \frac{1}{\sqrt{7}}\cos t\,dt \\ 0 \le x \le \pi/2 \end{array} \right| = \frac{1}{\sqrt{7}}\int \sqrt{1-\sin^2 x}\,\cos x\,dx = \frac{1}{\sqrt{7}}\int |\cos x|\cdot \cos x\,dx =$$
$$= \frac{1}{\sqrt{7}}\int \cos^2 x\,dx = \frac{1}{\sqrt{7}}\int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2\sqrt{7}} + \frac{1}{2\sqrt{7}}\int \cos 2x\,dx = \frac{x}{2\sqrt{7}} + \frac{\sin 2x}{4\sqrt{7}} =$$
$$= \frac{\arcsin(\sqrt{7}t)}{2\sqrt{7}} + \frac{\sin x\cos x}{2\sqrt{7}} = \frac{\arcsin(\sqrt{7}t)}{2\sqrt{7}} + \frac{\sqrt{7}t\sqrt{1-7t^2}}{2\sqrt{7}} = \frac{\arcsin(\sqrt{7}t)}{2\sqrt{7}} + \frac{t\sqrt{1-7t^2}}{2} + C$$

2. a) $y' + 4y = 4$ linjär $\Rightarrow$ I.F: $I(x) = e^{\int 4dx} = e^{4x}$

$e^{4x}y' + 4e^{4x}y = 4e^{4x}$

$(e^{4x}y)' = 4e^{4x}$

$e^{4x}y = 4\int e^{4x}dx = e^{4x} + C$

$y(x) = 1 + Ce^{-4x}$

b) $(1-x)^2 y' = y+1$, $y(1) = -1$

$\frac{dy}{y+1} = \frac{dx}{(1-x)^2}$, $y \neq -1$!

$\ln|y+1| = \frac{1}{1-x} + C$

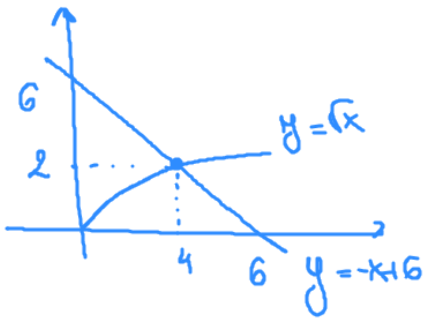
$|y+1| = e^{\frac{1}{1-x}+C} = Ce^{\frac{1}{1-x}}$

$|y+1| = \begin{cases} y+1 & \text{om } y > -1 \\ -y-1 & \text{om } y < -1 \end{cases}$

$y(x) = -1 + Ce^{1/(1+x)}$ om $y \neq -1$

$y(1) = -1 \Rightarrow y' = 0 \Rightarrow$ det är konstant lösning

3.



$$A=\int_0^4 \sqrt{x}\,dx + \int_4^6 (-x+6)\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 - \frac{x^2}{2}\Big|_4^6 + 6x\Big|_4^6$$

4. $g''\left(\frac{\pi}{6}\right) = ?$, $g(y) = \int_0^y f(x)\,dx$, $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$

Analysens huvudsats $\Rightarrow$ $g'(y) = \frac{d}{dy}\int_0^y f(x)\,dx = f(y)$

$$g''(y) = f'(y) = \frac{d}{dy}\int_0^{\sin y}\sqrt{1+t^2}\,dt = \sqrt{1+\sin^2 y}\cdot(\sin y)' = \cos y\sqrt{1+\sin^2 y}$$

$$g''\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6}\sqrt{1+\sin^2\frac{\pi}{6}} = \frac{\sqrt{3}}{2}\cdot\sqrt{1+\frac{1}{4}} = \frac{\sqrt{15}}{4}$$

5. $y(x) + \int_1^x y(t)\,dt = 2e^{2x}$ / derivera

$\frac{d}{dx}\int_1^x y(t)\,dt = y(x)$

$y' + y = 4e^{2x}$

I.F. $I(x) = e^{\int dx} = e^x$

$e^x y' + e^x y = 4e^{3x}$

$(e^x y)' = 4e^{3x}$

$e^x y = 4\int e^{3x}dx = \frac{4}{3}e^{3x} + C$

$y(x) = \frac{4}{3}e^{2x} + Ce^{-x}$

6. a) $\int_{-1}^{2} \frac{dx}{x^2-2x-3} = \underbrace{\lim_{a\to -1+}} \int_{a}^{2} \frac{dx}{x^2-2x-3} = \lim_{a\to -1+}\left(-\frac{1}{4}\int_{a}^{2}\frac{dx}{x+1} + \frac{1}{4}\int_{a}^{2}\frac{dx}{x-3}\right) =$

$= \lim_{a\to -1+}\left(-\frac{1}{4}\ln 3 + \underbrace{\frac{1}{4}\ln(a+1)}_{\to -\infty} + \cancel{\frac{1}{4}\ln|2-3|}^{\,0} - \cancel{\frac{1}{4}\ln|a-3|}^{\,\ln 4}\right) = \longrightarrow$ divergent

$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)} = \frac{(A+B)x-3A+B}{(x+1)(x-3)} = -\frac{1}{4}\cdot\frac{1}{x+1} + \frac{1}{4}\cdot\frac{1}{x-3}$

$\left.\begin{aligned} x^1 &: 0 = A+B \\ x^0 &: 1 = -3A+B \end{aligned}\right\} \Rightarrow A = -\frac{1}{4},\ B = \frac{1}{4}$

6. b) $\int_{0}^{\infty} \frac{1+e^{-x}}{x}\,dx = \infty$ divergent

$\frac{1+e^{-x}}{x} > \frac{1}{x}$ $\quad (e^{-x} > 0)$ $\Rightarrow \int_{0}^{\infty} \frac{1+e^{-x}}{x}\,dx > \int_{0}^{\infty} \frac{1}{x}\,dx = \infty$

$\int_{0}^{\infty} \frac{dx}{x} = \infty$ divergent

Jämförelse satsen
(skriva satsen!)

7. $y'' + 3y' + 2y = x(1+e^x)$

(h) $r^2 + 3r + 2 = 0 \Rightarrow r_{12} = \frac{-3 \pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases} \Rightarrow y_h(x) = C_1 e^{-x} + C_2 e^{-2x}$

(P1) $y_p(x) = (Ax+B)e^x$

$y_p'(x) = Ae^x + (Ax+B)e^x$

$y_p''(x) = Ae^x + Ae^x + (Ax+B)e^x$

$\underline{2Ae^x} + (Ax+B)e^x + \underline{3Ae^x} + 3(Ax+B)e^x + 2(Ax+B)e^x = xe^x$

$5A + 6Ax + 6B = x$

$\left.\begin{matrix} 5A + 6B = 0 \\ 6A = 1 \end{matrix}\right\} \Rightarrow A = \frac{1}{6}, B = -\frac{5}{36} \Rightarrow y_p(x) = \left(\frac{x}{6} - \frac{5}{36}\right)e^x$

(P2) $y_p(x) = Ax + B$

$y_p'(x) = A \quad , y_p''(x) = 0$

$3A + 2Ax + 2B = x$

$\left.\begin{matrix} 2A = 1 \\ 3A + 2B = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} A = \frac{1}{2} \\ B = -\frac{3}{4} \end{matrix}$

$y_p(x) = \frac{1}{2}x - \frac{3}{4}$

$h + p_1 + p_2 \Rightarrow y(x) = C_1 e^{-x} + C_2 e^{-2x} + \left(\frac{x}{6} - \frac{5}{36}\right)e^x + \frac{x}{2} - \frac{3}{4}$

8. $(Ae^{2x}+1)\cos x + Be^{2x}\sin x = \underbrace{(A\cos x + B\sin x)e^{2x}} + \cos x$

$\downarrow$ y_h $\qquad$ $\downarrow$ y_p

$\alpha = 2,\ \beta = 1 \Rightarrow r_{12} = 2 \pm i$

$(r-r_1)(r-r_2) = 0$

$(r-2-i)(r-2+i) = 0$

$r^2 - 2r - \cancel{ir} - 2r + 4 + \cancel{2i} + \cancel{ir} - \cancel{2i} + 1 = 0$

$r^2 - 4r + 5 = 0 \longrightarrow y'' - 4y' + 5y = 0 \longrightarrow$

$y = y_h + y_p$

$r_{12} \to y_h = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$r_{12} = \alpha \pm i\beta$

$\cos x$ ār y_p

$(\cos x)'' - 4(\cos x)' + 5(\cos x) =$

$= -\cos x + 4\sin x + 5\cos x =$

$= 4\cos x + 4\sin x$

$\downarrow$

$y'' - 4y' + 5y = 4\cos x + 4\sin x$

9. $y(t)$ vattendjupet , $y(0) = 3m$

$$\frac{dy}{dt} = -k\sqrt{y} \quad , \quad k > 0$$ Obs. minus tecken (vattendjupet minskar).

$$\frac{dy}{\sqrt{y}} = -k\,dt$$

$$2\sqrt{y} = -kt + C$$

$$y(t) = \left(-\frac{kt}{2} + C\right)^2$$

$$y(0) = 3 = C^2 \quad \Rightarrow \quad y(t) = \left(\sqrt{3} - \frac{kt}{2}\right)^2$$

10. Finns i Blandade uppgifter.