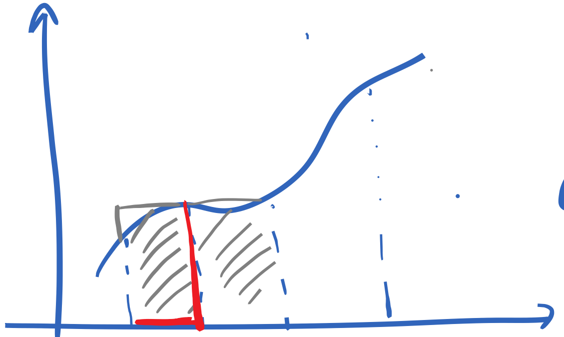
Blandade uppgifter
MVE545

Kapitel 5.1, uppgift 24 och 25

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$$

och vi har att $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\cdot \Delta x$

eftersom



$a \quad \Delta x \quad x_1$

$f(x_1) = f(a+\Delta x)$

$= f(a+\frac{3}{n})$

$f(x_2) = f(a+2\Delta x)$

$f(x_3) = f(a+3\Delta x)$

$\vdots$

$f(x_n) = f(b) = f(a+n\Delta x)$

$i=1$ ger oss x_1

$i=n$ ger oss $x_n = b$

$\frac{3}{n}$ ger oss Δx

$\Rightarrow n\cdot\frac{3}{n} = 3 = n\cdot\Delta x =$ längden av $[a,b]$

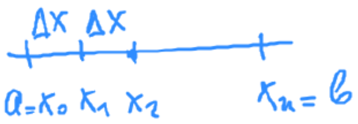
$\sqrt{1+\frac{3i}{n}} \Rightarrow a=1 \rightarrow b=4$

$\Rightarrow \int_1^4 \sqrt{1+x}\,dx \Rightarrow$ området under $\sqrt{1+x}$, $x\in[1,4]$

$\lim \sum \Delta x f(a+i\Delta x)$

25.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$



$x_1 = a + \Delta x$

$x_2 = a + \Delta x + \Delta x = a + 2\Delta x$

$x_i = a + i\Delta x$

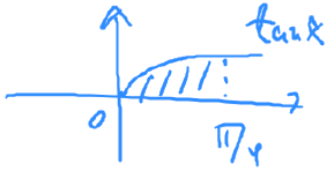
$i=1: \tan \frac{\pi}{4n}$

$i=n: \frac{n\pi}{4n} = \frac{\pi}{4}$

$a=0, \; b=\frac{\pi}{4}, \quad [0, \frac{\pi}{4}]$

$$\frac{\pi}{4n} = \Delta x = \frac{b-a}{n}$$

$$\Rightarrow \int_0^{\pi/4} \tan x \, dx$$



$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

$$\sum_{k=3}^{6} k = 3+4+5+6$$

$$\sum_{k=0}^{\infty} e^k = e^0 + e^1 + e^2 + \dots$$

$f(a+i\Delta x)$

$\sqrt{1+\frac{3i}{n}}$

5.3.

17. $$\frac{d}{dx}\left(\int_{\sqrt{x}}^{\pi/4} \theta \tan\theta \, d\theta\right) = -\sqrt{x}\tan(\sqrt{x})\cdot(\sqrt{x})' = -\cancel{\sqrt{x}}\tan(\sqrt{x})\cdot\frac{1}{2\cancel{\sqrt{x}}} = -\frac{1}{2}\tan(\sqrt{x})$$

$$\frac{d}{dx}\left(\int_{\pi/4}^{\sqrt{x}} \theta \tan\theta \, d\theta\right) = \frac{1}{2}\tan(\sqrt{x})$$

$$\frac{d}{dx}\left(\int_{\sqrt{x}}^{2x} \theta \tan\theta \, dx\right) = 2x\tan(2x)\cdot(2x)' - \sqrt{x}\tan(\sqrt{x})\cdot(\sqrt{x})'$$

$$= 4x\tan(2x) - \frac{1}{2}\tan(\sqrt{x})$$

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föreläsning

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

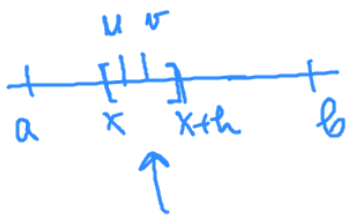
Analysens huvudsats

f kont. $[a,b]$

$g(x) = \int_a^x f(t)\,dt$

$\Rightarrow g$ kont. $[a,b]$

der. (a,b)



<u>Bevis:</u> $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{1}{h}\int_x^{x+h} f(t)\,dt \overset{\text{||}}{=} f(x)$

$$g(x+h)-g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt =$$

$$= \cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt - \cancel{\int_a^x f(t)\,dt} =$$

$$= \int_x^{x+h} f(t)\,dt$$

f kont. på $[a,b] \Rightarrow f$ kont. på $[x, x+h]$

$\Rightarrow \exists\, u, v \in [x, x+h]$, $f(u) = m$, $f(v) = M$

$m \leq f(t) \leq M$, $\forall t \in [x, x+h]$

$$\int_x^{x+h} m\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M\,dt$$

$$mh \leq \int_x^{x+h} f(t)\,dt \leq Mh \quad /:h$$

$$f(u) = m \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq M = f(v)$$

$h \to 0$, $x+h \to x \Rightarrow u \to x$, $v \to x$

$$\lim_{h\to 0} f(u) = f(x) = \lim_{h\to 0} f(v)$$

■

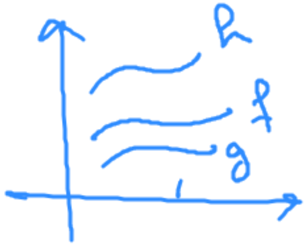
5.2 , 90

① a) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

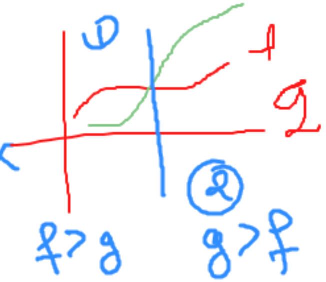
$\underset{g(x)}{-|f(x)|} \le f(x) \le \underset{h(x)}{|f(x)|}$ } sats

$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$

$$\boxed{\begin{array}{c} |u| \le n \\ -n \le u \le n \end{array}}$$



$\Rightarrow \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$



$-|2| \le 2 \le |2|$ $\quad f(x) = 2$

$-|f(x)| \le f(x) \le |f(x)|$, $\forall x \in D_f$

$g(x) = -|f(x)|$, $h(x) = |f(x)|$

$g(x) \le f(x) \le h(x)$

sats $\Rightarrow \int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$

$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$

$|x| \le 4$

$(-4 \le x \le 4)$

b) $$\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \overset{(a)}{\leq} \int_0^{2\pi} |f(x)\sin 2x|\,dx \leq \int_0^{2\pi} |f(x)|\cdot\underbrace{|\sin 2x|}_{\leq 1}\,dx \leq \int_0^{2\pi} |f(x)|\,dx$$

$$|xy| \leq |x|\cdot|y|$$

$$0 \leq |\sin 2x| \leq 1$$

$$-1 \leq \sin\alpha \leq 1,\ \forall\alpha$$

$$\int f(x)dx = \int f(t)dt$$

$$\int f(2x)dx \neq \int f(x)dx$$

Exempel 2: $\int_0^4 f(x)dx = 10$, $\int_0^2 f(2x)dx = ?$

$$\int_0^2 f(2x)dx = \left| \begin{array}{l} 2x = t \\ dx = \frac{dt}{2} \\ x=0, t=0,\ x=2, t=4 \end{array} \right| = \int_0^4 f(t)\cdot\frac{dt}{2} = \frac{1}{2}\int_0^4 f(t)dt = \frac{1}{2}\cdot 10 = 5$$

Anlysens huvudsats (bevis)

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$g(x) = \int_a^x f(t)dt \xrightarrow{\text{SATS}} g'(x) = f(x) \Rightarrow F(x) = g(x) + C \quad \text{primitiv funktion till } f$$

$a < x < b$, f, g kontinuerliga

$x \to a^+$, $x \to b^-$ $\Rightarrow [a, b]$

$$g(a) = \int_a^a f(t)dt = 0$$

$$F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) + C - 0 - C = g(b) = \int_a^b f(t)dt$$

Exempel: $f(x) = x^2 + x + 4$, $g(x) = \sin x + 5$, $[1, 5]$, $A = ?$

$$A = \int_1^5 (f(x) - g(x))dx$$

Exempel

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

① $\cos x = u$

$du = -\sin x\,dx$

$dx = -\dfrac{du}{\sin x}$

② $\cos x = t$

$x = \arccos t$

$dx = -\dfrac{dt}{\sqrt{1-t^2}}$

$x=0,\ t=1$

$x=\pi/2,\ t=0$

$(\arcsin t)' =$

$$\int_0^{\pi/2} f(\cos x)dx = \int_1^0 f(t)\cdot\left(-\frac{dt}{\sqrt{1-t^2}}\right) =$$

$$= -\int_1^0 \frac{f(t)}{\sqrt{1-t^2}}\,dt = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt$$

Exempel: $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$

5.5 / 69

$$\int_e^{e^4} \underbrace{\frac{dx}{x}}_{=dt}\,\underbrace{\frac{1}{\sqrt{\ln x}}}_{=\sqrt{t}} = \left| \begin{matrix} t=\ln x, & dt = \frac{dx}{x} \\ x=e, t=1; & x=e^4, t=4 \end{matrix} \right| = \int_1^4 \frac{dt}{\sqrt{t}} = \int_1^4 t^{-1/2}\,dt = \frac{t^{1/2}}{1/2}\Big|_1^4 = 2\left(\sqrt{4}-1\right) =$$

$$= 2(2-1) = 2$$

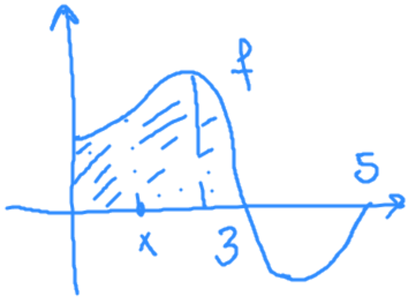
71. $$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \left| \begin{matrix} t = e^z + z \\ dt = e^z + 1 \\ z=0,\ t=e^0=1 \\ z=1,\ t=e+1 \end{matrix} \right| = \int_1^{e+1} \frac{dt}{t} = \ln t \Big|_1^{e+1}$$

5.4 / 1 od 9

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C \quad / \text{deriviram}$$

$$\int \underline{v(v^2+2)}^2\,dv = \left| \begin{matrix} t = v^2+2 \\ dt = 2v\,dv \end{matrix} \right| = \dots$$

5.3 /36



$$g(x) = \int_0^x f(t)\,dt$$

på vilket intervall är $g(x)$ växande?

$[0,3]$

5.5/55

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \left| \begin{array}{l} t = 1+7x \;,\; dt = 7dx \\ x=0, t=1 \;;\; x=1, t=8 \end{array} \right|$$

$$= \int_1^8 \frac{1}{7}\sqrt[3]{t}\,dt = \frac{1}{7}\left[\frac{t^{4/3}}{4/3}\right]_1^8 =$$

$$= \frac{3}{28}\left(8^{4/3}-1\right) = \frac{3}{28}(16-1) = \frac{45}{28}$$

$$8^{4/3} = (2^3)^{4/3} = 2^{3\cdot\frac{4}{3}} = 2^4 = 16$$

Exempel 1: $|x^2-4| = \begin{cases} x^2-4 & \text{om } x^2 \geq 4 \\ 4-x^2 & \text{om } x^2 < 4 \end{cases} = \begin{cases} x^2-4 & \text{om } x \geq 2 \\ 4-x^2 & \text{om } x \in [0,2) \end{cases}$

$$\int_0^3 |x^2-4|\,dx = \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \ldots$$

Exempel 2: $\sqrt{x} \geq 0$, $|\sqrt{x}-1| = \begin{cases} \sqrt{x}-1 & \text{om } \sqrt{x} \geq 1 \\ 1-\sqrt{x} & \text{om } 0 \leq \sqrt{x} < 1 \end{cases}$

$$\int_0^3 |\sqrt{x}-1|\,dx = \int_0^1 \qquad + \int_1^3$$

Exempel 3: $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \left| \begin{array}{l} t = x^2+1,\ dt = 2x\,dx \\ x^3\,dx = x^2 \cdot x\,dx = (t-1)\frac{dt}{2} \\ x^2 = t-1 \end{array} \right| = \int \frac{t-1}{\sqrt{t}} \cdot \frac{dt}{2} =$

$$= \frac{1}{2}\int \left(t^{1/2} - t^{-1/2}\right) dt$$

$$= \frac{1}{2}\left(\frac{t^{3/2}}{3/2} - \frac{t^{1/2}}{1/2}\right) =$$

Exempel 4: $\int_0^1 x^2\cos x\,dx \leq \frac{1}{3}$, $|\cos x| \leq 1$, $\cos x \leq 1 \Rightarrow x^2\cos x \leq x^2 \cdot 1$

$$\int_0^1 x^2\cos x\,dx \leq \int_0^1 1\cdot x^2\,dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Exempel 5: f kontinuerlig, $\int_0^2 f(x)\,dx = 6$, beräkna $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$

$$\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta = \left|\begin{array}{l} t=\sin\theta\ ,\ dt=\cos\theta\,d\theta \\ \theta=0,\ t=0 \\ \theta=\pi/2,\ t=1 \end{array}\right| = \int_0^1 f(2t)\,dt = \left|\begin{array}{l} u=2t\ ,\ du=2dt \\ t=0,\ u=0 \\ t=1,\ u=2 \end{array}\right|$$

$$= \frac{1}{2}\int_0^2 f(u)\,du = \frac{1}{2}\cdot 6 = 3$$

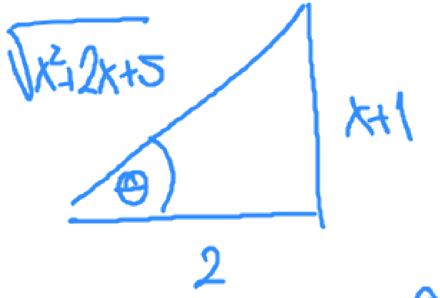
7.3 / 23

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \left| \begin{array}{l} x^2+2x+5 = (x+1)^2+2^2 \\ x+1 = 2\tan\theta \;,\; -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ dx = 2\,\frac{d\theta}{\cos^2\theta} \end{array} \right| = \int \frac{2\,d\theta}{\cos^2\theta\sqrt{4\tan^2\theta+4}} =$$

$$= \int \frac{\cancel{2}\,d\theta}{\cos^2\theta \cdot \cancel{2} \cdot \underbrace{\sqrt{\frac{\sin^2\theta}{\cos^2\theta}+1}}_{\parallel}} = \int \frac{d\theta}{\cos^2\theta \cdot \frac{1}{\cos\theta}} = \int \frac{d\theta}{\cos\theta} \overset{\text{Sidan 503}}{=} \ln\left|\frac{1}{\cos\theta} + \tan\theta\right| =$$

$$\sqrt{\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}} = \sqrt{\frac{1}{\cos^2\theta}} =$$

$$= \frac{1}{|\cos\theta|} = \frac{1}{\cos\theta}$$



$$\cos\theta = \frac{2}{\sqrt{x^2+2x+5}}$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| =$$

$$= \ln\left|\sqrt{x^2+2x+5} + x+1\right| - \ln 2 + C$$

$$= \ln\left|\sqrt{x^2+2x+5} + x+1\right| + C$$

annan lösning på nästa sida

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta \, d\theta}{\cos^2\theta} = \int \frac{\cos\theta \, d\theta}{1-\sin^2\theta} = \left| \begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \end{array} \right| = \int \frac{du}{1-u^2} =$$

$$\frac{1}{1-u^2} = \frac{A}{1+u} + \frac{B}{1-u} = \frac{A(1-u)+B(1+u)}{(1+u)(1-u)} = \frac{A+B+u(B-A)}{(1+u)(1-u)} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$

$$\left.\begin{array}{l} u: \ 0 = B-A \\ u^0: \ 1 = A+B \end{array}\right\} \Rightarrow \left.\begin{array}{l} A = B \\ A+B = 1 \end{array}\right\} \Rightarrow A = B = \frac{1}{2}$$

$$= \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u} = \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|$$

$$\sin\theta = \frac{x+1}{\sqrt{x^2+2x+5}}$$

$$= \frac{1}{2}\ln\left| \frac{1 + \frac{x+1}{\sqrt{x^2+2x+5}}}{1 - \frac{x+1}{\sqrt{x^2+2x+5}}} \right| + C$$

7.3/27

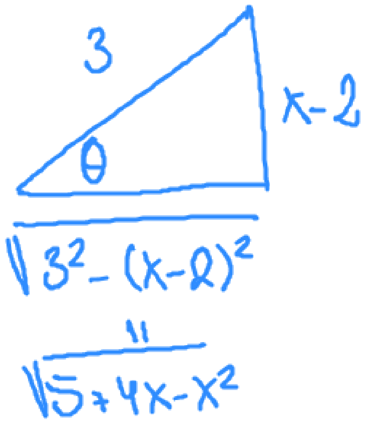
$$\int \sqrt{5+4x-x^2}\,dx = \left| \begin{array}{l} 5+4x-x^2 = 9\underline{-4+4x-x^2} = 9-(4-4x+x^2) = 3^2-(x-2)^2 \\ x-2 = 3\sin\theta \,,\; dx = 3\cos\theta\, d\theta \,,\; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array} \right| =$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \sqrt{\underbrace{9-9\sin^2\theta}_{9(1-\sin^2\theta)=9\cos^2\theta}} \cdot 3\cos\theta\, d\theta = \int 3\cos\theta \cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\int (1+\cos 2\theta)\, d\theta =$$

$$= \frac{9}{2}\int d\theta + \frac{9}{2}\int \cos 2\theta\, d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta = \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta$$

dela på 2



$$\sin\theta = \frac{x-2}{3}$$

$$\cos\theta = \frac{\sqrt{5+4x-x^2}}{3}$$

$$= \frac{9}{2}\arcsin\frac{x-2}{3} + \frac{\cancel{9}}{2} \cdot \frac{x-2}{\cancel{3}} \cdot \frac{\sqrt{5+4x-x^2}}{\cancel{3}}$$

$$= \frac{9}{2}\arcsin\frac{x-2}{3} + \frac{x-2}{2} \cdot \sqrt{5+4x-x^2} + C$$

7.3/25

$$\int x^2\sqrt{3+2x-x^2}\,dx = \left|\begin{array}{l} 3+2x-x^2 = 3-(x^2-2x+1)+1 = \overset{2^2}{\overset{\shortparallel}{4}}-(x-1)^2 \\ x-1 = 2\sin\theta \ , \ dx = 2\cos\theta\, d\theta \ , \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ x = 1+2\sin\theta \ , \ x^2 = 1+4\sin\theta+4\sin^2\theta \\ \sqrt{4-4\sin^2\theta} = 2\cos\theta \end{array}\right| =$$

$$= \int (1+4\sin\theta+4\sin^2\theta)\cdot \underbrace{2\cos\theta\cdot 2\cos\theta}_{4\cos^2\theta}\, d\theta = \int (1+4\sin\theta+4\sin^2\theta)\cdot 4\cos^2\theta\, d\theta =$$

$$= 4\int \cos^2\theta\, d\theta + 16\int \sin\theta\cos^2\theta\, d\theta + 16\int \sin^2\theta\cos^2\theta\, d\theta =$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\frac{1}{4}\sin^2 2\theta$$

$$= 4\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$\cos\theta = t \ , \ -\sin\theta\, d\theta = dt$$

7.4 / 13

$$\int \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int \left(1-\frac{4}{x^3-2x^2}\right)dx = \int dx - \int \frac{4\,dx}{x^3-2x^2} = x + \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{dx}{x-2}$$

$$\frac{4}{x^3-2x^2} = \frac{4}{x^2(x-2)} = \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)} =$$

$$= \frac{A(x^2-2x)+B(x-2)+Cx^2}{x^2(x-2)} = \frac{x^2(A+C)+x(-2A+B)-2B}{x^2(x-2)} = -\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x-2}$$

$$\left.\begin{aligned} x^2&: \quad 0 = A+C \\ x&: \quad 0 = -2A+B \\ x^0&: \quad 4 = -2B \end{aligned}\right| \Rightarrow \left.\begin{aligned} &B=-2 \\ &A = \frac{B}{2} = -1 \\ &-1+C=0 \end{aligned}\right| \Rightarrow \begin{aligned} &A=-1 \\ &B=-2 \\ &C=1 \end{aligned}$$

7.3/2

$$\int x^3\sqrt{9-x^2}\,dx = \left|\begin{matrix} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{matrix}\right| = \int (3\sin\theta)^3 \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta =$$

$$= \int 3^3 \sin^3\theta \cdot 3\cos\theta \cdot 3\cos\theta\, d\theta = 3^5\int \sin^3\theta\cos^2\theta\, d\theta = 3^5\int \sin\theta(1-\cos^2\theta)\cos^2\theta\, d\theta$$

$$= \left|\begin{matrix} t = \cos\theta \\ dt = -\sin\theta d\theta \end{matrix}\right| = -3^5\int(1-t^2)t^2\,dt = -3^5\int(t^2-t^4)\,dt = \dots \quad +C$$

$$9-9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta \qquad \sin^3\theta = \sin\theta\cdot\sin^2\theta = \underline{\sin\theta}(1-\cos^2\theta)$$

$$\sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Ordinära differentialekvationer

9.3/43 (sidan 606)

Exempel 1:

$$\frac{dC}{dt} = r - kC, \quad k>0$$

a) $C(t_0) = C_0$

b) $C_0 < \frac{r}{k}$, $\lim_{t\to\infty} C(t)$

$\frac{dy}{dt} = 0 \to y = \text{konstant}$
eller någon annan
lösning → jämviktspunkt
(equilibrium point)

a) $\frac{dC}{dt} = r - kC$

$\frac{dC}{r-kC} = dt$, $r - kC \neq 0$

$-\frac{1}{k} \ln|r-kC| = t + D$, D konstant

$\ln|r-kC| = -kt + D$

$r - kC = e^{-kt+D} = De^{-kt}$

$kC = r + De^{-kt}$

$C = \frac{1}{k}(r + De^{-kt})$

$C(t_0) = C_0 \Rightarrow C_0 = \frac{1}{k}(r + \cancel{De^{-k\cdot 0}}) \Rightarrow D = kC_0 - r$

$C = \frac{r}{k} + \frac{kC_0 - r}{k} e^{-kt}$

$\frac{dC}{dt} = r - kC = 0$

$r - kC = 0$

$\Downarrow$

$C = r/k$, $C' = 0$

$\Downarrow$

$0 = r - k \cdot \frac{r}{k} = 0$

$\Downarrow$

$C = r/k$

lösning

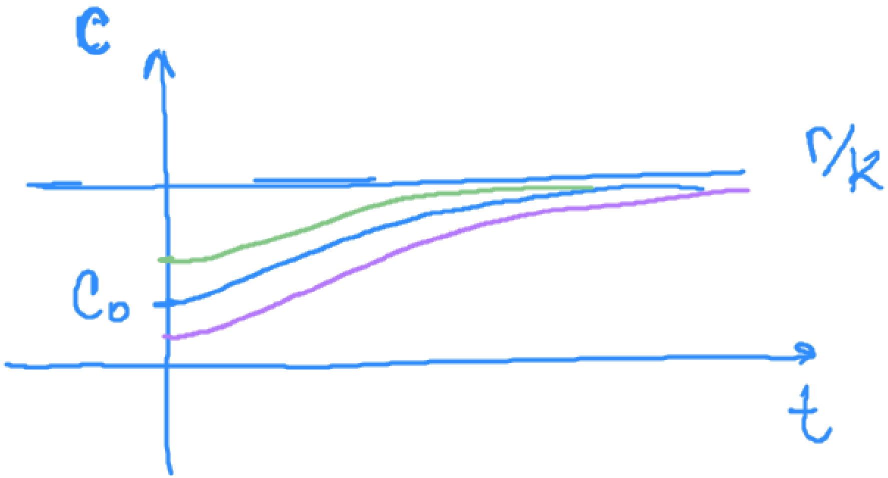
b) $C(t) = \frac{r}{k} + \frac{kC_0 - r}{k} e^{-kt}$, $C(t_0) = t_0$

(jämviktspunkt : $C = \frac{r}{k}$
konstant lösning)

$C(t) = \frac{r}{k} + \frac{D}{k} e^{-kt}$ allmän lösning

$C_0 < \frac{r}{k}$, $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \left(\frac{r}{k} + \frac{D}{k} e^{-kt}\right) = \frac{r}{k}$

$t \to \infty$, $-kt \to -\infty$, $e^{-kt} \to 0$



$\frac{dy}{dt} = r - ky = 0$ (?) $f' = 0 \to f$ konst.

$\downarrow$ $y = r/k$

för att hitta jämviktspunkter löser vi

$$\frac{dy}{dt} = 0$$

om möjligt. Många ODE har ingen jämvikts

9.3/13

Exempel 2:

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -3$$

$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

$$-3 = -\sqrt{0^2 + C}$$

$$\Downarrow$$

$$C = 9$$

$$\Downarrow$$

$$y = -\sqrt{x^2 + 9}$$

$$\sqrt{x} = a, \quad a > 0$$

9.5/37 (sida 626)

Exempel 3:

$$P' = kP\left(1 - \frac{P}{M}\right) \quad \text{ej linjär ODE}$$

$$Z = \frac{1}{P} \Rightarrow P = \frac{1}{Z}, \quad P' = -\frac{Z'}{Z^2}$$

$$-\frac{Z'}{Z^2} = k\cdot\frac{1}{Z}\left(1 - \frac{1}{ZM}\right) = \frac{k}{Z} - \frac{k}{Z^2M}$$

$$Z' = -Z^2\left(\frac{k}{Z} - \frac{k}{Z^2M}\right)$$

$$Z' = -kZ + \frac{k}{M}$$

$$\boxed{Z' + kZ = \frac{K}{M}}$$

I.F.

$$P(t) = \frac{1}{Z(t)}, \quad P'(t) = \left(\frac{1}{Z(t)}\right)' = -\frac{1}{Z^2(t)}\cdot Z'(t) = -\frac{Z'(t)}{Z^2(t)}$$

Z(t) är funktion (som P) som beror på t

$$\text{I.F.}: \quad I(t) = e^{\int k\,dt} = e^{kt}$$

$$\underbrace{e^{kt}Z' + ke^{kt}Z}_{=(e^{kt}Z)'} = \frac{k}{M}e^{kt}$$

$$(e^{kt}Z)' = \frac{k}{M}e^{kt}$$

$$e^{kt}Z = \int \frac{k}{M}e^{kt}\,dt$$

$$e^{kt}Z = \frac{1}{M}e^{kt} + C \quad !$$

$$\boxed{Z(t) = \frac{1}{M} + Ce^{-kt}}$$

$$P(t) = \left(\frac{1}{M} + Ce^{-kt}\right)^{-1}$$

eller

$$P(t) = \frac{1}{Z(t)} = \frac{1}{\frac{1}{M} + Ce^{-kt}} = \frac{M}{1 + CMe^{-kt}} = \frac{M}{1 + Ce^{-kt}}$$

б) $k=0.1$, $M=250$, $P(0)=15$

$P(100) = ?$

$\lim\limits_{t\to\infty} P(t) = ?$

$$P(t) = \frac{M}{1+Ce^{-kt}}$$

$$\lim_{t\to\infty} \frac{M}{1+Ce^{-kt}} =$$

$$15 = \frac{M}{1+C\underbrace{e^{-k0}}_{=1}} = \frac{M}{1+C} = \frac{250}{1+C}$$

$$\Rightarrow 1+C = \frac{250}{15} \Rightarrow C = \frac{250}{15} - 1$$

$$P(t) = \frac{250}{1+\left(\frac{250}{15}-1\right)e^{-0.1t}}$$

$$P(100) = \frac{250}{1+\left(\frac{250}{15}-1\right)e^{-10}}$$

Parametervariation

Exempel 1: $y'' + y = \tan x \quad , \quad 0 < x < \frac{\pi}{2}$

(h) $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_h(x) = C_1 \cos x + C_2 \sin x \quad , \quad C_1, C_2$ konstanter

(p) $y_p = u_1(x)\cos x + u_2(x)\sin x \quad , \quad u_1(x), u_2(x)$ är funktioner

$y_p' = u_1'\cos x - u_1 \sin x + u_2' \sin x + u_2 \cos x = (u_1'\cos x + u_2'\sin x) - u_1 \sin x + u_2 \cos x$

$u_1'\cos x + u_2'\sin x = 0$ vi väljer så

$y_p' = -u_1 \sin x + u_2 \cos x$

$y_p'' = -u_1'\sin x - u_1 \cos x + u_2'\cos x - u_2 \sin x$

$y_p'' + y_p = -u_1'\sin x - \cancel{u_1\cos x} + u_2'\cos x - \cancel{u_2 \sin x} + \cancel{u_1 \cos x} + \cancel{u_2 \sin x} =$

$= -u_1'\sin x + u_2'\cos x = \tan x$

Exempel 2: $\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right)$, $P(t) = \frac{1}{z(t)}$ (9.5/37)

(9.5/38) $\frac{dP}{dt} = k(t)P\left(1-\frac{P}{M(t)}\right)$

$P(t) = \frac{1}{z(t)}$, $P'(t) = -\frac{z'(t)}{z^2(t)}$ $\Rightarrow -\frac{z'(t)}{z^2(t)} = k(t)\frac{1}{z(t)}\left(1-\frac{1}{z(t)M(t)}\right)$

$$z'(t) = -k(t)z(t)\left(1-\frac{1}{z(t)M(t)}\right)$$

$$z'(t) = -k(t)z(t) + \frac{k(t)}{M(t)}$$

$$z'(t) + k(t)z(t) = \frac{k(t)}{M(t)}$$

I.F: $I(t) = e^{\int k(t)dt}$

$$z'(t) + k(t) z(t) = \frac{k(t)}{M(t)} \quad , \text{I.F} \quad e^{\int k(t)dt}$$

$$\left(e^{\int k(t)dt} z(t) \right)' = \frac{k(t)}{M(t)} e^{\int k(t)\,dt}$$

$$e^{\int k(t)dt} z(t) = \int \frac{k(t)}{M(t)} e^{\int k(s)ds} dt + C$$

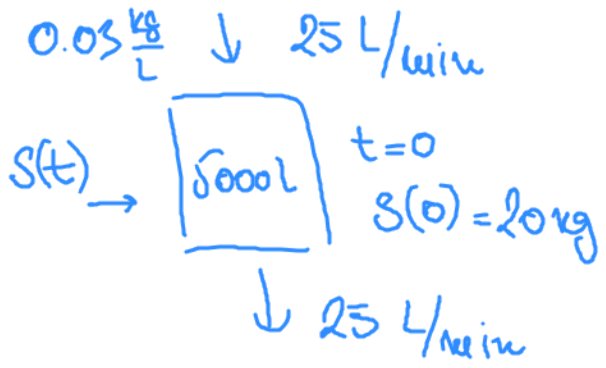
$$z(t) = e^{-\int k(t)dt} \left(\int \frac{k(t)}{M(t)} e^{\int k(s)ds} dt + C \right)$$

$$P(t) = \frac{1}{z(t)} = \frac{e^{\int k(t)dt}}{\int \frac{k(t)}{M(t)} e^{\int k(s)ds} + C}$$

Exempel 3 (604):

$s(0) = 20\,\text{kg}$, $5000\,\text{L}$

$0.03\,\text{kg/L}$, $25\,\text{L/min}$



$$\frac{ds}{dt} = 0.03\,\frac{\text{kg}}{\cancel{\text{L}}} \cdot 25\,\frac{\cancel{\text{L}}}{\text{min}} - \frac{S(t)}{5000\,\text{L}}\,\text{kg} \cdot 25\,\frac{\text{L}}{\text{min}}$$

$$\frac{ds}{dt} = 0.75 - \frac{s(t)}{200} \qquad (\text{kg/min})$$

$$\frac{ds}{dt} + \frac{1}{200}\,s(t) = 0.75 \quad , \quad \text{I.F.} \quad I(t) = e^{\int \frac{dt}{200}} = e^{t/200}$$

$$\left(e^{t/200}\,s(t)\right)' = 0.75\,e^{t/200}$$

$$e^{t/200}\,s(t) = 0.75 \int e^{t/200}\,dt + C$$

$$S(t) = e^{-t/200}\left(0.75 \int e^{t/200}\,dt + C\right)$$

$S(0) = 20 \Rightarrow C =$

$S(60) =$

Repetition

Exempel 1: $G(x)=\int_1^{x^2}(t^2-3t)\,dt$, $G'(x)=?$

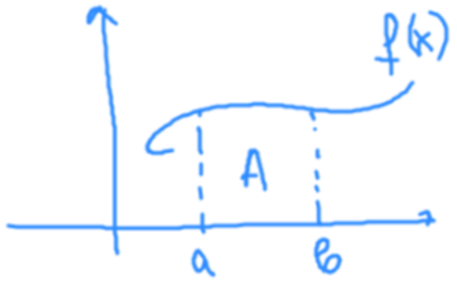
$$G'(x)=\left((x^2)^2-3(x^2)\right)2x$$

Obs! Skriva Analysens huvudsats.

Exempel 2: $G(x)=\int_x^{x^3}\tan(t^2)\,dt$, $G'(x)=?$

$$G'(x)=\tan(x^3)^2\cdot 3x^2-\tan(x^2)\cdot 1=3x^2\tan(x^6)-\tan(x^2)$$

Generaliserade integraler:

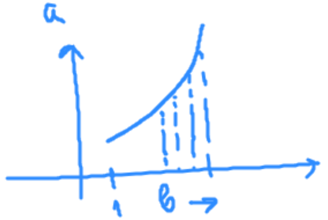


1. $[a,\infty)$, $(-\infty,b]$, $(-\infty,\infty)$
2. f icke kontinuerlig på $[a,b]$

om $A<\infty$ → integral är konvergent
om $A=\infty$ → " divergent

Exempel 3: $\displaystyle\int_1^\infty \frac{dx}{x^2} = \boxed{\lim_{b\to\infty} \int_1^b \frac{dx}{x^2}} = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$

$\displaystyle\int_a^b f(x)dx$



$\displaystyle\int_1^\infty \frac{dx}{x^2} = 1 \quad \to$ konvergent

Exempel 4: $\displaystyle\int_1^\infty \frac{dx}{3x} = \frac{1}{3}\lim_{b\to\infty}\int_1^b \frac{dx}{x} = \frac{1}{3}\lim_{b\to\infty} \ln x \Big|_1^b = \frac{1}{3}\lim_{b\to\infty}(\ln b - \cancel{\ln 1}^{\,0}) =$

$\displaystyle = \frac{1}{3}\lim_{b\to\infty} \ln b = \infty \quad \to$ divergent

Exempel 5: $\int_{-2}^{\infty} \sin x\,dx = \lim_{b\to\infty} \int_{-2}^{b} \sin x\,dx = \lim_{b\to\infty} (\cos 2 - \cos b) = \cos 2 - \boxed{\lim_{b\to\infty} \cos b}$

divergent

$\lim_{x\to\pm\infty} \sin x = ?$

$\lim_{x\to\pm\infty} \cos x = ?$

finns inte!

Exempel 6: $\int_{0}^{3} \frac{dx}{\sqrt{3-x}} = \boxed{\lim_{b\to 3-} \int_{0}^{b} \frac{dx}{\sqrt{3-x}}} = \left| \begin{array}{l} 3-x = u \\ dx = -du \\ x=0,\ u=3 \\ x=b,\ u=3-b \end{array} \right| = \lim_{b\to 3-} \int_{3}^{3-b} \frac{du}{\sqrt{u}} =$

$= \lim_{b\to 3-} \left(-2\sqrt{u}\Big|_{3}^{3-b}\right) = \lim_{b\to 3-} \left(2\sqrt{3} - \cancel{2\sqrt{3-b}}\right) = 2\sqrt{3} \to$ konvergent

$\cancel{2\sqrt{3-b}} \to 0$

Exempel 7: $\int_{-2}^{3}\frac{1}{x^3}\,dx = \int_{-2}^{0}\frac{1}{x^3}\,dx + \int_{0}^{3}\frac{1}{x^3}\,dx = \lim_{b\to 0-}\int_{-2}^{b}\frac{dx}{x^3} + \lim_{a\to 0+}\int_{a}^{3}\frac{dx}{x^3} =$

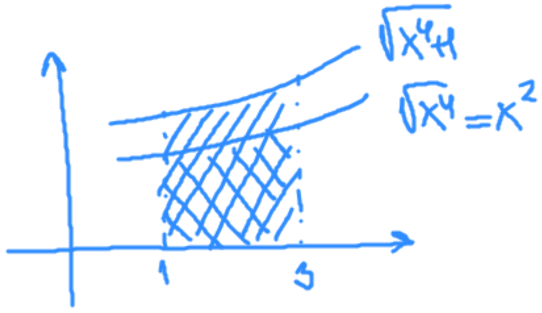
$$= \lim_{b\to 0-}\left(-\frac{1}{2x^2}\Big|_{-2}^{b}\right) + \lim_{a\to 0+}\left(-\frac{1}{2x^2}\Big|_{a}^{3}\right) =$$

Exempel 8: Bevisa att: $\int_{1}^{3}\sqrt{x^4+1}\,dx \geq \frac{26}{3}$

$$\sqrt{x^4+1} \geq \sqrt{x^4} = x^2 \quad\Rightarrow\quad \int_{1}^{3}\sqrt{x^4+1}\,dx \geq \int_{1}^{3}x^2\,dx = \frac{x^3}{3}\Big|_{1}^{3} = \frac{26}{3}$$

$x^2+1 \geq x^2$

$x^2+1 \geq x^2-1$



1. approximera integrand
2. så enkelt som möjligt

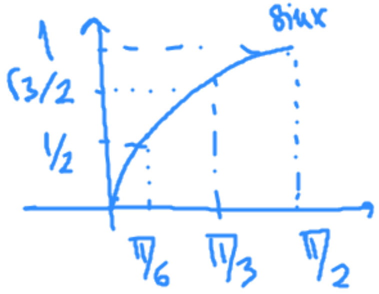
Exempel 9: $\int_0^{\pi/2} x\sin x\,dx \le \frac{\pi^2}{8}$

$x\sin x$, $[0, \pi/2] \Rightarrow x \le \frac{\pi}{2}$ och $\sin x \le 1 \Rightarrow x\cdot\sin x \le \frac{\pi}{2}\cdot 1$ (ok men löser inte uppgift)

$$\int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\cdot 1\,dx = \frac{x^2}{2}\Big|_0^{\pi/2} = \frac{\pi^2}{8}$$

$\sin x \le 1$

Exempel 10: $\frac{\pi}{12} \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \frac{\pi\sqrt{3}}{12}$



$I = \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

längden: $\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} = l$

① $\sin x$ växande på $I \Rightarrow \sin\frac{\pi}{6} \le \sin x \le \sin\frac{\pi}{3}$

$$\frac{1}{2} \le \sin x \le \frac{\sqrt{3}}{2}$$

$$\int_{\pi/6}^{\pi/3} \frac{1}{2}\,dx \le \int_{\pi/6}^{\pi/3} \sin x\,dx \le \int_{\pi/6}^{\pi/3} \frac{\sqrt{3}}{2}$$

$$\frac{\pi}{12} \le \text{—}\,||\,\text{—} \le \frac{\sqrt{3}\pi}{12}$$

② $l\cdot\frac{1}{2}$

$l\cdot\frac{\sqrt{3}}{2}$

Exempel 1: $y'' + 2y' + y = e^{-x} \arctan x$

$r^2 + 2r + 1$
$(r+1)^2$

$$\left.\begin{aligned} y' + y &= z \quad , \quad z = z(x) \\ y'' + y' &= z' \end{aligned}\right\} \Rightarrow y'' + 2y' + y = \underline{z' + z = e^{-x} \arctan x}$$

I.F. $I(x) = e^{\int dx} = e^{x}$

$z'e^{x} + ze^{x} = \arctan x$

$(ze^{x})' = \arctan x$

$ze^{x} = \int \arctan x \, dx \overset{P.I.}{=} x\arctan x - \frac{1}{2}\ln(x^2+1) + C$

$z(x) = e^{-x}\left(x\arctan x - \frac{1}{2}\ln(x^2+1) + C\right)$

$y' + y = e^{-x}\left(x\arctan x - \frac{1}{2}\ln(x^2+1) + C\right)$

I.F: $I(x) = e^{x}$

$y'e^{x} + ye^{x} = x\arctan x - \frac{1}{2}\ln(x^2+1) + C$

$(ye^{x})' = x\arctan x - \frac{1}{2}\ln(x^2+1) + C$

$ye^{x} = \int\left(x\arctan x - \frac{1}{2}\ln(x^2+1) + C\right)dx$

Exempel 2: $f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$, på vilket intervall är $f(x)$ växande ?

$f'(x) = (1-x^2)e^{x^2} > 0$ om $x \in (0,1)$

Exempel 3: $0 \le \int_1^\infty \frac{1}{\sqrt{x}e^x}\,dx \le \int_1^\infty \frac{dx}{e^x} = \int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left(\frac{1}{e} - \cancelto{0}{e^{-b}}\right)$

$= \frac{1}{e} < \infty$

$x > 1 \Rightarrow \sqrt{x} > 1 \,,\, e^x > 1 \Rightarrow \sqrt{x}e^x > e^x \Rightarrow \frac{1}{\sqrt{x}e^x} < \frac{1}{e^x}$

$f \le g \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx < \infty$

$x = 4 : \sqrt{4} = 2 > 1 \,,\, \sqrt{4}\cdot e^4 = 2e^4 > e^4$

$\int_0^\infty \frac{1}{\sqrt{x}e^x}\,dx = \int_0^1 \frac{1}{\sqrt{x}e^x}\,dx + \int_1^\infty \frac{1}{\sqrt{x}e^x}\,dx =$

$0 \leftarrow$

$a \to 0+$

$= \lim_{a\to 0+} \int_a^1 \frac{1}{\sqrt{x}e^x}\,dx +$

Exempel 4:

$$y' + xy = x - x^2y' - xy$$

$$(1+x^2)y' + 2xy = x$$

$$y' + \boxed{\frac{2x}{1+x^2}}\, y = \frac{x}{1+x^2}$$

$$\text{I.F}: \quad I(x) = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$$

$$\left((1+x^2)y\right)' = x$$

$$(1+x^2)y = \frac{x^2}{2} + C$$

$$y = \frac{x^2+C}{2(1+x^2)}$$

Exempel 5: $\int_{-1}^{2}(x-2|x|)dx = \int_{-1}^{0}(x-2(-x))dx + \int_{0}^{2}(x-2x)dx =$

$|x| = \begin{cases} x & \text{om } x \geq 0 \\ -x & \text{om } x < 0 \end{cases}$ $\quad = \int_{-1}^{0}(x+2x)dx + \int_{0}^{2}(-x)dx$

Exempel 6: $\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}dx = \left| \begin{matrix} t=e^{x} \\ dt=e^{x}dx \end{matrix} \right| = \int \frac{t\,dt}{t^2+3t+2} =$

$e^{2x}dx = \underbrace{e^{x}}_{=t} \cdot \underbrace{e^{x}dx}_{=dt}$

$\frac{t}{t^2+3t+2} = \frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} = \frac{A(t+2)+B(t+1)}{(t+1)(t+2)}$

$1\cdot t = At + 2A + Bt + B \Rightarrow \begin{matrix} 1 = A+B \\ 0 = 2A+B \end{matrix}$

Exempel 1: Bevisa att: $\int_0^1 x^a (1-x)^b \, dx = \int_0^1 (1-x)^a x^b \, dx$

? ?

$1-x = t$

$x = 1-t$

$-dx = dt$

$x=0,\ t=1$

$x=1,\ t=0$

$$\Rightarrow \int_0^1 x^a (1-x)^b \, dx = -\int_1^0 (1-t)^a t^b \, dt = \int_0^1 (1-t)^a t^b \, dt =$$

$$= \int_0^1 (1-x)^a x^b \, dx$$

$x = \sin t$

$dx = \cos t \, dt$

$x=0,\ t=0$

$x=1,\ t=\frac{\pi}{2}$

Exempel 2: f kontinuerlig, $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$

$u = \cos x$, $x = \arccos u \to dx = \ldots$

$du = -\sin x \, dx$

$dx = -\frac{du}{\sin x} = -\frac{du}{\sqrt{1-\cos^2 x}} = -\frac{du}{\sqrt{1-u^2}}$

$x=0,\ u=1$

$x=\frac{\pi}{2},\ u=0$

!

$$VS = -\int_1^0 f(u)\frac{du}{\sqrt{1-u^2}} = \int_0^1 f(u)\frac{du}{\sqrt{1-u^2}}$$

HS på samma sätt

$v = \sin x$

Exempel 3: $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx \quad , \quad u = \pi - x$

$u = \pi - x$
$du = -dx$
$x = 0,\ u = \pi$
$x = \pi,\ u = 0$

$$\Rightarrow \int_0^{\pi} x f(\sin x)\,dx \overset{u=\pi-x}{=} -\int_{\pi}^{0} (\pi - u) f(\sin(\pi - u))\,du = \int_0^{\pi} (\pi - u) f(\sin(\pi - u))\,du =$$

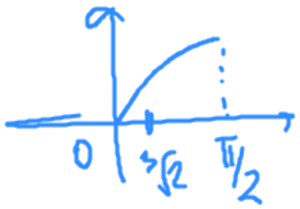
$$= \pi \int_0^{\pi} f(\sin u)\,du - \int_0^{\pi} u f(\sin u)\,du \overset{x=u}{=} \pi \int_0^{\pi} f(\sin x)\,dx - \int_0^{\pi} x f(\sin x)\,dx$$

$$2\int_0^{\pi} x f(\sin x)\,dx = \pi \int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

Sats: $f(x) = f(-x) \Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$f(x) = -f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx = 0$



$\pi \approx 3.14$

$\sqrt{2} \approx 1.4$

$\sqrt{3} \approx 1.73$

Exempel 4: $f(x) = \sin\sqrt[3]{x}$ udda funkt (?)

$$0 \leq \int_{-2}^{3} \sin\sqrt[3]{x}\,dx \leq 1 \quad (?)$$

$$f(-x) = \sin\sqrt[3]{(-x)} = \sin(-\sqrt[3]{x}) = -\sin\sqrt[3]{x} = -f(x)$$

$$\int_{-2}^{3} \sin\sqrt[3]{x}\,dx = \underbrace{\int_{-2}^{2} \sin\sqrt[3]{x}\,dx}_{=0} + \int_{2}^{3} \sin\sqrt[3]{x}\,dx = \int_{2}^{3} \sin\sqrt[3]{x}\,dx$$

$$2 \leq x \leq 3$$

$$\sqrt[3]{2} \leq \sqrt[3]{x} \leq \sqrt[3]{3}$$

$$\boxed{0 \leq \sin\sqrt[3]{x} \leq 1}$$

$$\int_{2}^{3} 0\cdot dx \leq \int_{2}^{3} \sin\sqrt[3]{x}\,dx \leq \int_{2}^{3} 1\cdot dx$$

$$0 \leq \int_{2}^{3} \sin\sqrt[3]{x}\,dx \leq 1$$

Exempel 5: För vilka p är $\int_0^1 \frac{dx}{x^p}$ konvergent?

$$\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0+} \int_a^1 \frac{dx}{x^p} = \lim_{a\to 0+} \frac{x^{1-p}}{1-p}\Big|_a^1 = \frac{1}{1-p}$$

$$\left.\begin{array}{l} 1-p>0 \\ 0\le p<1 \\ p<0 \end{array}\right| \Rightarrow p<1$$

$p=2$

$$\lim_{a\to 0+} \frac{x^{1-2}}{-1}\Big|_a^1 = -\lim_{a\to 0+} \frac{1}{x}\Big|_a^1 = -\lim_{a\to 0+} \left(1-\cancel{\frac{1}{a}}_{\infty}\right) = \infty$$

Exempel 6:

$$\int_0^\infty \frac{dx}{x(\ln x)^p} = \int_0^1 \frac{dx}{x(\ln x)^p} + \int_1^\infty \frac{dx}{x(\ln x)^p} = \lim_{a\to 0+} \int_a^1 \frac{dx}{x(\ln x)^p} + \lim_{b\to\infty} \int_1^b \frac{dx}{x(\ln x)^p}$$

30.5.2020

3. $y'' - 2y' + y = f(x)$, a) $f(x) = x$, b) $f(x) = e^{-x}$, c) $f(x) = e^{x}$, d) $f(x) = xe^{x}$

(h) $r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Rightarrow r_{12} = 1 \Rightarrow y_h(x) = C_1 e^{x} + C_2 x e^{x}$

a) $y_p(x) = Ax + B$

b) $y_p(x) = Ae^{-x}$

c) $y_p(x) = x^2 e^{x}$ $\quad [(Ax^2 + Bx + C)e^{x}]$

d) $y_p(x) = x^2 e^{x}$ $\quad [(Ax^2 + Bx + C)e^{x}]$

Ex: $y'' + 3y' = x$

$y_p(x) = (Ax + B)x$

$y_p(x) = Ax^2 + Bx + C$

Ex: $y'' = 3x$

$y'(x) = \int 3x\,dx$

Exempel: $$y'' - 2y' + y = -\frac{1}{x^2}e^x$$

(h) $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_{12} = 1$

$$y_h(x) = C_1 e^x + C_2 x e^x$$

(p) $y(x) = z(x)e^x$

$$y'(x) = z'(x)e^x + z(x)e^x = (z' + z)e^x$$

$$y''(x) = (z'' + 2z' + z)e^x$$

$$(z'' + \cancel{2z'} + \cancel{z} - \cancel{2z'} - \cancel{2z} + \cancel{z})\cancel{e^x} = -\frac{1}{x^2}\cancel{e^x}$$

$$z'' = -\frac{1}{x^2}$$

$$z'(x) = -\int \frac{dx}{x^2} = \frac{1}{x} + C$$

$$z(x) = \int \left(\frac{1}{x} + C\right)dx = \ln|x| + Cx + D$$

$$y_p(x) = (\ln|x| + Cx + D)e^x$$

$$y = y_h + y_p$$

Exempel: $0 \le \int_5^{10} \frac{x^2}{x^4+x^2+1}\,dx \le \frac{1}{10}$

$$\frac{x^2}{x^4+x^2+1} \ge 0 \Rightarrow \int_5^{10} \frac{x^2}{x^4+x^2+1}\,dx \ge 0$$

$$\frac{x^2}{x^4+x^2+1} \le \frac{x^2}{x^4} = \frac{1}{x^2}$$

$$\int_5^{10} \frac{x^2}{x^4+x^2+1}\,dx \le \int_5^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_5^{10} = \frac{1}{10}$$

$$\int_{-5}^{0} \frac{dx}{x+3} = \int_{-5}^{-3} \frac{dx}{x+3} + \int_{-3}^{0} \frac{dx}{x+3} = \lim_{b\to -3^-} \int_{-5}^{b} \frac{dx}{x+3} +$$

$$+ \lim_{a\to -3^+} \int_{a}^{0} \frac{dx}{x+3} = \lim_{b\to -3^-} \ln|x+3|\Big|_{-5}^{b} + \lim_{a\to -3^+} \ln|x+3|\Big|_{a}^{0}$$

Exempel: $\int x^3 \cos(x^4+2)\,dx$

Exempel: $\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} + \int_{0}^{1}$

$$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x}$$

Exempel:

$$\int_{-5}^{5} \frac{|x|}{x^2+3x}\,dx = \int_{-5}^{0} \frac{-x}{x^2+3x}\,dx +$$

$$+ \int_0^5 \frac{x}{x^2+3x}\,dx = -\int_{-5}^{0} \frac{1}{x+3}\,dx$$

$$+ \int_0^5 \frac{1}{x+3}\,dx = \boxed{-\int_{-5}^{-3} + \int_{-3}^{0}} + \int_0^5$$