1. a) $\int_{-2}^{2} \frac{|x|}{x^2-12x}\,dx = \int_{-2}^{0} \frac{-x}{x^2-12x}\,dx + \int_{0}^{2} \frac{x}{x^2-12x}\,dx =$

$= -\int_{-2}^{0} \frac{dx}{x-12} + \int_{0}^{2} \frac{dx}{x-12} = -\ln|x-12|\,\Big|_{-2}^{0} + \ln|x-12|\,\Big|_{0}^{2} =$

$= -\ln 12 + \ln 14 + \ln 10 - \ln 12 = \ln\frac{140}{144} = \ln\frac{35}{36}$

b) $\int \arctan\frac{1}{t}\,dt = \left| \begin{array}{l} u = \arctan\frac{1}{t}, \quad du = \frac{1}{1+(\frac{1}{t})^2}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2}\,dt \\ dv = dt, \quad v = t \end{array} \right|$

$= t\arctan\frac{1}{t} + \int \frac{t}{1+t^2}\,dt = t\arctan\frac{1}{t} + \frac{1}{2}\ln(1+t^2) + C$

c) $\int \frac{x+1}{x^2+x+\frac{5}{4}}\,dx = \int \frac{x+1}{(x+\frac{1}{2})^2+1}\,dx = \left| \begin{array}{l} x+\frac{1}{2} = t \\ dx = dt \end{array} \right| = \int \frac{t+\frac{1}{2}}{t^2+1}\,dt$

$= \int \frac{t}{t^2+1}\,dt + \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\ln(t^2+1) + \frac{1}{2}\arctan t =$

$= \frac{1}{2}\ln\left(x^2+x+\frac{5}{4}\right) + \frac{1}{2}\arctan\left(x+\frac{1}{2}\right) + C$

2. a) $y' + 4y = e^x$

i.f.: $I(x) = e^{\int 4dx} = e^{4x}$

$(e^{4x}y)' = e^{5x}$

$e^{4x}y(x) = \frac{1}{5}e^{5x} + C$

$y(x) = \frac{1}{5}e^{x} + Ce^{-4x}$

b) $(x-1)^2 y' = y^2+1 \quad , \; y(0)=0$

$$\frac{dy}{y^2+1} = \frac{dx}{(x-1)^2}$$

$$\arctan y = -\frac{1}{x-1} + C$$

$y(0)=0 \Rightarrow 0 = 1+C \Rightarrow C=-1 \Rightarrow y = \tan\left(-\frac{1}{x-1} - 1\right)$

c) $y'' - 4y' - 5y = x \quad , \; y(0) = y(1) = 0$

ⓐ $r^2 - 4r - 5 = 0 \Rightarrow r_{12} = \frac{4 \pm 6}{2} = \begin{cases} 5 \\ -1 \end{cases} \Rightarrow y_a(x) = C_1 e^{5x} + C_2 e^{-x}$

ⓟ $y_p(x) = Ax + B$

$y_p'(x) = A \quad , \; y_p''(x) = 0$

$-4A - 5Ax - 5B = x$

$\left.\begin{aligned} -4A - 5B &= 0 \\ -5A &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -1/5 \\ B &= 4/25 \end{aligned} \Rightarrow y_p(x) = -\frac{x}{5} + \frac{4}{25}$

$\Rightarrow y(x) = y_a(x) + y_p(x) = C_1 e^{5x} + C_2 e^{-x} - \frac{x}{5} + \frac{4}{25}$

$\left.\begin{aligned} y(0) &= C_1 + C_2 + \frac{4}{25} = 0 \\ y(1) &= C_1 e^5 + C_2 e^{-1} - \frac{1}{5} + \frac{4}{25} = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 + C_2 &= -\frac{4}{5} \\ C_1 e^5 + C_2 e^{-1} &= \frac{1}{25} \end{aligned}$

$\Rightarrow C_1 = \frac{e+20}{25(e^6-1)} \quad , \; C_2 = -\frac{4}{5} - \frac{e+20}{25(e^6-1)}$

$\Rightarrow y(x) = \frac{e+20}{25(e^6-1)} e^{5x} - \left(\frac{4}{5} + \frac{e+20}{25(e^6-1)}\right) e^{-x} - \frac{x}{5} + \frac{4}{25}$

t=-x^2, dt=-2xdx
x=0, t=0; x=2, t=-4

3. $$A = \int_0^2 (x+1-xe^{-x^2})\,dx = \frac{x^2}{2}\Big|_0^2 + x\Big|_0^2 - \int_0^2 xe^{-x^2}\,dx =$$

$$= 4 + \frac{1}{2}\int_0^{-4} e^t\,dt = 4 + \frac{1}{2}e^t\Big|_0^{-4} = 4 + \frac{1}{2}e^{-4} - \frac{1}{2} = \frac{7}{2} + \frac{e^{-4}}{2}$$

4. a) $$\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} - 4$$

enligt Analysens huvudsats $\frac{d}{dx}\left(\int_a^x \frac{f(t)}{t^2}\,dt\right) = \frac{1}{\sqrt{x}}$

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x\sqrt{x}$$

För att hitta konstanten a:

$$\int_a^x \frac{t\sqrt{t}}{t^2}\,dt = 2\sqrt{x} - 4$$

$$\int_a^x \frac{dt}{\sqrt{t}} = 2\sqrt{t}\Big|_a^x = 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x} - 4$$

$$2\sqrt{a} = 4 \Leftrightarrow a = 4$$

b) $$g'(x) = \frac{d}{dx}\int_{1-3x}^{1+2x} t\sin t\,dt = (1+2x)\sin(1+2x)\cdot(1+2x)' -$$

$$-(1-3x)\sin(1-3x)\cdot(1-3x)' = 2(1+2x)\sin(1+2x) +$$

$$+ 3(1-3x)\sin(1-3x)$$

c) Funktion $f(t) = \frac{t\arcsin t}{1+t^2}$ är kontinuerlig. Enligt Analysens huvudsats, $F(x)$ är primitiv funktion till $-\frac{x\arcsin x}{1+x^2}$ eftersom $F'(x) = -\frac{x\arcsin x}{1+x^2}$.

5. $f(-x) = \sin(-x)^3 = \sin(-x^3) = -\sin x^3 = -f(x)$

a) $$\int_{-\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{3}}} \sin x^3 dx = \underbrace{\int_{-\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{6}}} \sin x^3 dx}_{=0 \text{ enligt Sats}} + \int_{\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{3}}} \sin x^3 dx =$$

$$= \int_{\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{3}}} \sin x^3 dx$$

$$x \in \left[\sqrt[3]{\frac{\pi}{6}}, \sqrt[3]{\frac{\pi}{3}}\right] \Rightarrow x^3 \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \Rightarrow \sin x^3 \in \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

$\Rightarrow$ 1. $\sin x^3$ är positiv på intervall $\Rightarrow 0 \leq \int_{\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{3}}} \sin x^3 dx$

2. $\sin x^3 \leq \frac{\sqrt{3}}{2}$ för alla $x \in \left[\sqrt[3]{\frac{\pi}{6}}, \sqrt[3]{\frac{\pi}{3}}\right]$

$$\int_{\sqrt[3]{\frac{\pi}{6}}}^{\sqrt[3]{\frac{\pi}{3}}} \sin x^3 dx \leq \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{\pi}{3}} - \sqrt[3]{\frac{\pi}{6}}\right)$$

Från 1. och 2. följer (a).

b) $\int_{-\pi/4}^{\pi/4} (1-x^2)\sin x^3 dx = 0$ eftersom intergrand är en udda funktion

$g(x) = (1-x^2)\sin x^3 dx = \sin x^3 - x^2 \sin x^3 = f(x) - x^2 \sin x^3$

Vi vet att $f(x)$ är en udda funktion och vi kommer att visa att $h(x) = x^2 \sin x^2$ är också udda funktion:

$h(-x) = (-x)^2 \sin(-x)^3 = x^2 \sin(-x^3) = -x^2 \sin x^3 = -h(x)$

(P2) $y_{p2}(x) = Ae^x \,,\quad y_p'(x) = Ae^x \,,\quad y_p''(x) = Ae^x$

$Ae^x + Ae^x - 6Ae^x = e^x$

$-4A = 1 \Rightarrow A = -\frac{1}{4} \Rightarrow y_{p2}(x) = -\frac{1}{4}e^x$

(P3) $y_{p3}(x) = A\cos x + B\sin x$

$y_{p3}'(x) = -A\sin x + B\cos x$

$y_{p3}''(x) = -A\cos x - B\sin x$

$-A\cos x - B\sin x - A\sin x + B\cos x - 6A\cos x - 6B\sin x = \sin x$

$$\left.\begin{array}{l} -7A + B = 0 \\ -A - 7B = 1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} B = 7A \\ -50A = 1 \end{array}\right\} \Leftrightarrow \begin{array}{l} A = -1/50 \\ B = -7/50 \end{array}$$

$y_{p2}(x) = -\frac{1}{50}\cos x - \frac{7}{50}\sin x$

$y(x) = y_a(x) + y_{p1}(x) + y_{p2}(x) + y_{p3}(x)$

$y(x) = C_1 e^{2x} + C_2 e^{-3x} - \frac{x}{6} - \frac{1}{36} - \frac{1}{4}e^x - \frac{1}{50}\cos x - \frac{7}{50}\sin x$

6. a) $\int_{-\infty}^{0} \cos x \, dx = \lim_{a \to -\infty} \int_{a}^{0} \cos x \, dx = \lim_{a \to -\infty} (\sin 0 - \sin a) =$

$= -\lim_{a \to -\infty} \sin a$ divergent eftersom gränsvärden existerar inte

b) $\int_{0}^{\infty} \frac{1}{\sqrt{x}+e^x} dx \leq \int_{0}^{\infty} \frac{1}{e^x} dx$ eftersom $\sqrt{x}+e^x \geq e^x, \forall x \geq 0$

$\int_{0}^{\infty} \frac{1}{e^x} dx = \lim_{b \to \infty} \int_{0}^{b} e^{-x} dx = \lim_{b \to \infty} (e^0 - e^{-b}) = 1$

$\Rightarrow \int_{0}^{\infty} \frac{1}{e^x} dx$ konvergent

Enligt Jämförelsesats följer det att

$0 \leq \int_{0}^{\infty} \frac{1}{\sqrt{x}+e^x} dx \leq \int_{0}^{\infty} \frac{1}{e^x} dx < \infty$ konvergent.

7. $y'' + y' - 6y = x + e^x \sin x)$

(h) $r^2 + r - 6 = 0$

$r_{12} = \frac{-1 \pm 5}{2} = \begin{cases} 2 \\ -3 \end{cases} \Rightarrow y_h(x) = C_1 e^{2x} + C_2 e^{-3x}$

(p1) $y_{p1}(x) = Ax + B, \quad y_{p1}'(x) = A, \quad y_{p1}''(x) = 0$

$A - 6Ax - 6B = x$

$\left.\begin{array}{l} A - 6B = 0 \\ -6A = 1 \end{array}\right\} \Rightarrow A = -\frac{1}{6}, \; B = -\frac{1}{36}$

$y_{p1}(x) = -\frac{x}{6} - \frac{1}{36}$

8. $y'' - 2y' + y = \frac{e^x}{1+x^2}$

$y(x) = z(x)e^x$, $z(x)$ kontinuerlig , deriverbar
$z'(x)$ — " — — " —

$y'(x) = (z'(x) + z(x))e^x$
$y''(x) = (z''(x) + 2z'(x) + z(x))e^x$
$(z'' + 2z' + z - 2z' - 2z + z)e^x = \frac{e^x}{1+x^2}$
$z''(x) = \frac{1}{1+x^2}$
$z'(x) = \int \frac{dx}{1+x^2} = \arctan x + C$
$z(x) = \int \arctan x \, dx + Cx \overset{P.I}{=} x\arctan x - \frac{1}{2}\ln(1+x^2) +$
$+ Cx + D$

$y(x) = (x\arctan x - \frac{1}{2}\ln(1+x^2) + Cx + D)e^x$

9. $\frac{dT}{dt} = k(T - T_{ow})$, $T(0) = 80°C$, $T_{ow} = 20°C$, $k = -0.01$
$T(t_1) = 50°C$, $t_1 = ?$

$\frac{dT}{T - T_{ow}} = k\,dt$

$\ln|T - T_{ow}| = kt + C$

$|T - T_{ow}| = e^{kt + C} = Ce^{kt}$ $\Rightarrow$ $80°C - 20°C = 60 = C$

$T(t_1) = 50°C$ $\Rightarrow$ $50 - 20 = 60e^{-0.01t_1}$
$e^{-0.01t_1} = 1/2$
$-0.01t_1 = -\ln 2$
$t_1 = 100 \cdot \ln 2$