Tools in functional analysis

Let V be a vector space over $\mathbb{R}$.

<u>Def</u>: A mapping $\|\cdot\| : V \to \mathbb{R}$ is a norm if for all $v, w \in V$, $\lambda \in \mathbb{R}$

(i) $\|v\| = 0 \iff v = 0$

(ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$

(iii) $\|v+w\| \leq \|v\| + \|w\|$

Two norms are equivalent if

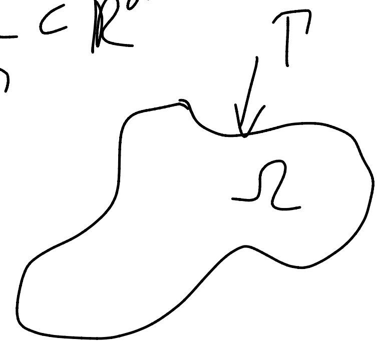
$$c\|\cdot\|_a \leq \|\cdot\|_b \leq C\|\cdot\|_a$$

<u>Def</u>:
* A linear mapping $L : V \to \mathbb{R}$ is called a Linear functional
* A bilinear form $a : V \times V \to \mathbb{R}$ is linear in both arguments.
* A bilinear form is symmetric if $a(v,w) = a(w,v)$ $\forall v,w \in V$ and positive if $a(v,v) > 0$ for

all non-zero $v \in V$.

* If both positive and symmetric it is an inner product or scalar product and it induces a norm $a(v,v)^{1/2} = \|v\|$.
* If a is a scalar product and $a(v,w)=0$ we say v,w are orthogonal.

Ex. $\begin{cases} -\nabla \cdot A \nabla u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \Gamma \end{cases}$



Multiplication with a test function v and $\int_\Omega$

$$L(v) := \int_\Omega f \cdot v \, dx = \int_\Omega -\nabla \cdot A \nabla u \cdot v \, dx = \int_\Omega A \nabla u \cdot \nabla v \, dx$$

$$=: a(u,v).$$

<u>Lemma</u> Let V be a vector space with an inner product $(\cdot,\cdot)$ and $\|\cdot\| = (\cdot,\cdot)^{1/2}$.

(i) Cauchy-Schwarz $|(v,w)| \le \|v\| \cdot \|w\|$

(ii) Pythagoras $\|v+w\|^2 = \|v\|^2 + \|w\|^2$, $(v,w)=0$

(iii) $\|\cdot\| = (\cdot,\cdot)^{1/2}$ defines a norm on V.

Proof. (i) let $t \in \mathbb{R}$

$$0 \leq \|v - tw\|^2 = (v - tw, v - tw) =$$

$$= \|v\|^2 - 2t(v,w) + t^2\|w\|^2$$

Now let $t = \frac{(v,w)}{\|w\|^2}$ assuming $w \neq 0$

$$0 \leq \|v\|^2 - 2\frac{(v,w)^2}{\|w\|^2} + \frac{(v,w)^2}{\|w\|^2} = \|v\|^2 - \frac{(v,w)^2}{\|w\|^2}$$

$$|(v,w)|^2 \leq \|v\|^2 \cdot \|w\|^2 .$$

(ii) $\|v+w\|^2 = (v+w, v+w) = \|v\|^2 + \|w\|^2$

(iii) Is $\|\cdot\| = (\cdot,\cdot)^{1/2}$ a norm?

$$\|v+w\|^2 = (v+w, v+w) = \|v\|^2 + 2(v,w) + \|w\|^2$$

$$\leq \|v\|^2 + 2\|v\|\cdot\|w\| + \|w\|^2$$

$$\leq (\|v\| + \|w\|)^2 .$$

A linear functional is bounded if

$$|L(v)| \leq C\|v\| , \quad \forall v \in V$$

The set of linear functionals

forms a vectorspace called the dual space V^* $\quad \|L\|_{V^*} = \sup_{v \in V, v \neq 0} \frac{|L(v)|}{\|v\|}$

A bounded bilinear form: There exists $M > 0$ s.t. $|a(v,w)| \leq M \cdot \|v\| \cdot \|w\|$ $\quad v, w \in V$.

and coercive if there is $\alpha > 0$ s.t.

$$a(v,v) \geq \alpha \|v\|^2, \quad \forall v \in V$$

<u>Def</u>.

* A sequence $\{v_i\}_{i=1}^{\infty}$ in V is said to converge to $v \in V$ if $\|v_i - v\| \to 0$ as $i \to \infty$.
* If $\|v_i - v_j\| \to 0$ when $i,j \to \infty$ the sequence is Cauchy.
* A linear space is said to be complete if every Cauchysequence converges.

* A complete space with a norm

is a Banach space. A Banach space with inner product it is a Hilbert space.

* If V is a Banach space so is V^*.
* A subset of a Hilbert space $V_0 \subset V$ is closed if it contains all limits of sequences in V_0. A closed subspace of a Hilbert space is also a Hilbert space.

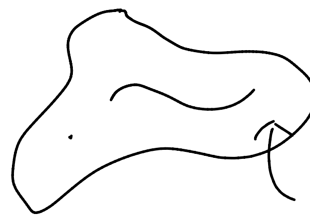
Thm A.1 Riesz

Let V be a Hilbert space with a scalar product $(\cdot,\cdot)$ and a linear functional L on V. Then there exists a unique $u \in V$ such that $L(v) = (u,v) \quad \forall v \in V$. Furthermore $\|L\|_{V^*} = \|u\|_V$.

Thm A.3 Lax-Milgram's Lemma

Let V be a Hilbert space and assume a coercive and bounded and L linear functional that is bounded. Then there is a unique $u \in V$ s.t. $\underline{a(u,v) = L(v)}$ $\forall v \in V$

L^p spaces



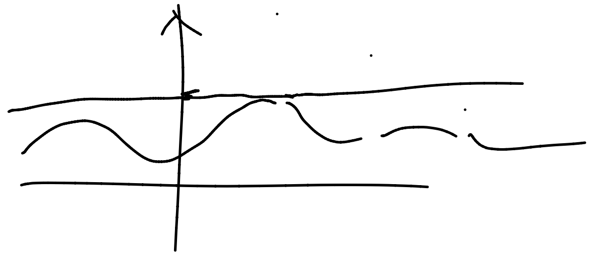
$\Omega \subset \mathbb{R}^d$

Def Let $\omega \subset \Omega$ such that $\int_\omega 1\,dx = 0$. Then ω is a null set. A statement that is true except on a null set is said to be true almost everywhere.

Let Ω be a bounded domain in $\mathbb{R}^d$.

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v|^p\,dx \right)^{1/p}, \quad 1 \le p < \infty$$

$$\|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| = \inf\{a\in\mathbb{R} : |v(x)| \le a \text{ almost every } x\in\Omega\}$$



We define the L^p spaces as

$$L^p(\Omega) = \{v : \|v\|_{L^p(\Omega)} < \infty\}.$$

<u>Thm</u> Let $v\in L^p(\Omega)$, $w\in L^q(\Omega)$ then

$$\|vw\|_{L^1(\Omega)} \le \|v\|_{L^p(\Omega)} \cdot \|w\|_{L^q(\Omega)}$$

$\frac{1}{p}+\frac{1}{q}=1$ Hölders inequality.

$$\|v+w\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)}$$

Minkowski's inequality.

<u>Proof</u>: Minkowski. $p=1$

$|v+w| \le |v|+|w|$. Assume $v+w$ is not zero everywhere and

$$|v+w|^p \le (|v|+|w|)\,|v+w|^{p-1}$$

$$\int_\Omega |v+w|^p\,dx \le \int_\Omega (|v|+|w|)(|v+w|)^{p-1}\,dx$$

$$\le \|v\|_{L^p(\Omega)} \cdot \| |v+w|^{p-1} \|_{L^q(\Omega)}$$

$$+ \|w\|_{L^p(\Omega)} \cdot \| |v+w|^{p-1} \|_{L^q(\Omega)}$$

$$\le \left(\|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)}\right) \cdot \left(\int_\Omega |v+w|^{(p-1)q}\,dx\right)^{1/q}$$

$$\le \left(\|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)}\right) \cdot \left(\int_\Omega |v+w|^p\,dx\right)^{1/q}$$

$$\left\{ \begin{array}{l} (p-1)q = (p-1)\left(\frac{1}{1-\frac{1}{p}}\right) = (p-1)\left(\frac{p}{p-1}\right) = p \\ \frac{1}{q} + \frac{1}{p} = 1 \quad \frac{1}{q} = 1 - \frac{1}{p} \end{array} \right\}$$

$\Rightarrow$

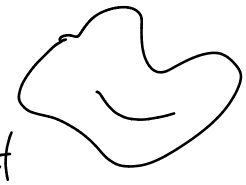
$$\left(\int_\Omega |v+w|^p\,dx\right)^{1-\frac{1}{q} = \frac{1}{p}} \le \|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)}$$

$$\|v+w\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)} \blacksquare$$

The Minkowski inequality and

$$\|\lambda v\|_{L^p(\Omega)} = |\lambda| \, \|v\|_{L^p(\Omega)}.$$

In order for $\|\cdot\|_{L^p(\Omega)}$ to be a norm



__Def__: Functions v and w that differs on a null set are said to be equivalent. We write

$$v = w \quad \underset{\text{almost}}{a} \cdot \underset{\text{everywhere}}{e}.$$

Now we have that $\|\cdot\|_{L^p(\Omega)}$ a norm and $L^p(\Omega)$ is a Banach space.

The special case $p=2$ $L^2(\Omega)$ is a Hilbert space $(v,w) = \int_\Omega v \cdot w \, dx$

$$(v,v)^{1/2} = \left(\int_\Omega v^2 \, dx\right)^{1/2} = \|v\|_{L^2(\Omega)}, \quad v \in L^2(\Omega)$$