Sobolev spaces

The weak derivative

Let $\Omega \subset \mathbb{R}^d$ bounded. Let $v \in C^1(\bar{\Omega})$.
Further let $C_0^1(\Omega) = \{ w \in C^1(\bar{\Omega}) : \text{supp} \subset \Omega \}$

$$\int_\Omega \frac{\partial v}{\partial x_i} \phi \, dx = - \int_\Omega v \cdot \frac{\partial \phi}{\partial x_i} dx , \quad i = 1, \dots, d$$

for $\phi \in C_0^1(\Omega)$.

We define a linear functional

$$L(\phi) := - \int_\Omega v \cdot \frac{\partial \phi}{\partial x_i} dx , \quad \forall \phi \in C_0^1(\Omega)$$

We call L the weak derivative of v defined for all $v \in L^1(\Omega)$.

If there is a function $w \in L^2(\Omega)$ such that $- \int_\Omega v \frac{\partial \phi}{\partial x_i} dx = \int_\Omega w \cdot \phi \, dx \quad \forall \phi \in C_0^1(\Omega)$

we say that the weak derivative belongs to $L^2(\Omega)$ and write $\frac{\partial v}{\partial x_i} = w$.

Ex: Let $v = |x|$ $\Omega = [-1, 1]$

Then $w = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

since

$$-\int_{-1}^{1} |x| \frac{\partial \phi}{\partial x}\, dx = +\int_{-1}^{0} x \frac{\partial \phi}{\partial x}\, dx - \int_{0}^{1} x \frac{\partial \phi}{\partial x}\, dx$$

$$= -\int_{-1}^{0} \phi\, dx + \cancel{[x\phi]_{-1}^{0}} + \int_{0}^{1} \phi\, dx - \cancel{[x\phi]_{0}^{1}}$$

$$= \int_{-1}^{1} w \phi\, dx \quad \forall \phi \in C_0^1([-1, 1])$$

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1\, \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_d} x_d},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_d)$

$|\alpha| = \sum_{i=1}^{d} \alpha_i$

We define

$$D^\alpha v(\phi) = (-1)^{|\alpha|} \int_\Omega v D^\alpha \phi\, dx, \forall \phi \in C_0^{|\alpha|}(\Omega)$$

If $D^\alpha v$ is bounded in $L^2(\Omega)$ we write $(\underline{D^\alpha v}, \phi) = (-1)^{|\alpha|}(v, D^\alpha \phi)$ $\forall \phi \in C_0^{|\alpha|}(\Omega)$

$$\left((v, w) = \int_\Omega v \cdot w\, dx \right)$$

Sobolev spaces

We let $H^k(\Omega) = \{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \; |\alpha| \le k \}$

$$(v,w)_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha v \cdot D^\alpha w \, dx$$

$$\|v\|^2_{H^k(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega (D^\alpha v)^2 dx$$

$$\|v\|^2_{H^1(\Omega)} = \int_\Omega v^2 + \sum_{j=1}^d \left(\frac{\partial v}{\partial x_j}\right)^2 dx =$$

$$= \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

$$\|v\|^2_{H^2(\Omega)} = \int_\Omega v^2 + \sum_{j=1}^d \left(\frac{\partial v}{\partial x_j}\right)^2 + \sum_{i,j=1}^d \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 dx$$

$$= \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|D^2 v\|^2_{L^2(\Omega)}$$

$$|v|^2_{H^k(\Omega)} = \sum_{|\alpha|=k} \int_\Omega (D^\alpha v)^2 dx$$

$H^k(\Omega)$ are Hilbert spaces.

It can be shown that $C^l(\bar{\Omega})$ is dense in $H^k(\Omega)$ for any $l \ge k$ if the $\partial\Omega$ is sufficiently smooth. For every $v \in H^k(\Omega)$ and every $\varepsilon > 0$ there is a

$w \in C^1(\bar{\Omega})$ such that $\|w - v\|_{H^k(\Omega)} < \varepsilon$.

The functions in $H^1(\Omega)$ can be discontinuous if $d \geq 2$.

See $v(x) = \log(-\log|x|)$, $\Omega = \{x \in \mathbb{R}^2 : |x| < \frac{1}{2}\}$.

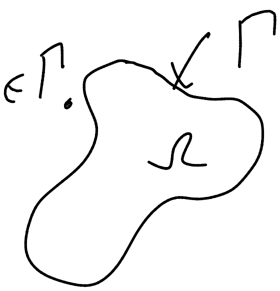
In 2D $r\,dr$ 1D dx.

For $H^2(\Omega)$ we have $\|v\|_{L^\infty(\Omega)} \leq C \|D^2 v\|_{L^2(\Omega)}$

Trace theorem

A function $v \in C^1(\bar{\Omega})$ is well defined on the boundary Γ of Ω

The trace $(\gamma v)(x) = v(x)$, $\forall x \in \Gamma$.

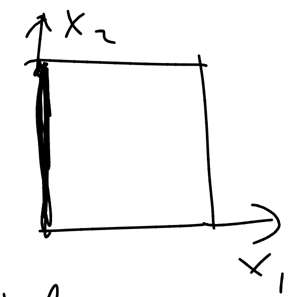


Thm A.4

Let $\Omega \subset \mathbb{R}^d$ bounded with smooth or polygonal boundary. The trace operator $\gamma : C^1(\bar{\Omega}) \to C(\Gamma)$ may be extended to $\gamma : H^1(\Omega) \to L^2(\Gamma)$.

$$\|\gamma v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}, \forall v \in H^1(\Omega)$$

Proof: Let $\Omega = [0,1]^2$.



First consider $v \in C^1(\bar{\Omega})$.

We use that

$$v(0,x_2) = v(x_1,x_2) + \int_{x_1}^{0} \frac{\partial v}{\partial x_1}(s,x_2)\,ds$$

$$|v(0,x_2)|^2 \le \left(|v(x_1,y_2)| + \left| \int_0^{x_1} \frac{\partial v}{\partial x_1}(s,y_2)\,ds \right| \right)^2 \le$$

$$\le \left\{ \begin{array}{l} (a+b)^2 \le 2a^2+2b^2 \\ 0 \le (a-b)^2 = a^2-2ab+b^2 \Rightarrow 2ab \le a^2+b^2 \end{array} \right\}$$

$$\le 2|v(x_1,x_2)|^2 + 2\left(\int_0^{x_1} 1 \cdot \left| \frac{\partial v}{\partial x_1}(s,x_2) \right| ds \right)^2$$

$$\le 2|v(x_1,x_2)|^2 + 2 \int_0^1 1^2\,dx \cdot \int_0^1 \left| \frac{\partial v}{\partial x_1}(s,x_2) \right|^2 ds$$

$$\int_0^1 |v(0,x_2)|^2\,dx_1 \le 2\int_0^1 |v(x_1,x_2)|^2\,dx_1 + 2\int_0^1 \left| \frac{\partial v}{\partial x_1}(s,x_2) \right|^2 ds$$

$$\int_0^1 |v(0,x_2)|^2\,dx_2 \le 2\|v\|^2_{L^2(\Omega)} + 2\|\nabla v\|^2_{L^2(\Omega)}$$

$$\le 2\|v\|^2_{H^1(\Omega)}$$

$$\|v\|^2_{L^2(\Gamma)} \le 8\|v\|^2_{H^1(\Omega)} \Rightarrow \|v\|_{L^2(\Gamma)} \le 8^{1/2}\|v\|_{H(\Omega)}$$

Let $v \in H^1(\Omega)$. Since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ there is a sequence $\{v_i\}_{i=1}^{\infty}$

in $C^1(\bar{\Omega})$ such that $\|v_i - v\|_{H^1(\Omega)} \to 0$ $i \to \infty$

This sequence is Cauchy in $H^1(\Omega)$ i.e.

$\|v_i - v_j\|_{H^1(\Omega)} \to 0$ as $i,j \to \infty$. Since $v_i - v_j \in C^1(\bar{\Omega})$

$\|\gamma v_i - \gamma v_j\|_{L^2(\Gamma)} \le C \|v_i - v_j\|_{H^1} \to 0$ $i,j \to \infty$

$\{\gamma v_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $L^2(\Gamma)$

i.e $\gamma v_i \to w \in L^2(\Gamma)$. let $\gamma v = w$

For any $\varepsilon_1 > 0$ $\exists n_1$ such that

$\|w - \gamma v_i\|_{L^2(\Gamma)} < \varepsilon_1$ $i > n_1$

For any $\varepsilon_2 > 0$ $\exists n_2$ s.t.

$C \|v - v_i\|_{H^1(\Omega)} < \varepsilon_2$ $i > n_2$

$\|w\|_{L^2(\Gamma)} \le \|\gamma v_i\|_{L^2(\Gamma)} + \|w - \gamma v_i\|_{L^2(\Gamma)}$

$\le C \|v_i\|_{H^1(\Omega)} + \varepsilon_1 \le C\|v\|_{H^1(\Omega)} + \varepsilon_1 + \varepsilon_2$

for $i > \max(n_1, n_2)$

$\therefore$ $\|w\|_{L^2(\Gamma)} \le C \|v\|_{H^1(\Omega)}$ $v \in H^1(\Omega)$

Given $v \in H^1(\Omega)$ assume $w_1 = \lim \gamma v_i$

$w_2 = \lim \gamma v_j'$

where $v_i \to v$ and $v_j' \to v$ in $H^1(\Omega)$.

and $w_1 \ne w_2$

For any $\varepsilon > 0$ there is an n

$$\|w_1 - w_2\|_{L^2(\Gamma)} \le \|w_1 - \gamma v_i\|_{L^2(\Gamma)} + \|\gamma(v_i - v_j')\|_{L^2(\Gamma)} + \|w_2 - v_j'\|_{L^2(\Gamma)}$$

$$\le 2\varepsilon + C\|v_i - v_j'\|_{H^1(\Omega)} \le$$

$$\le 2\varepsilon + C\|v_i - v\|_{H^1(\Omega)} + \|v_j' - v\|_{H^1(\Omega)}$$

$$\le 4\varepsilon$$ Contradiction i.e. $w_1 = w_2$

The construction is unique. ∎

The spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$

We let $H^1_0(\Omega) = \{ v \in H^1(\Omega) : \gamma v = 0 \}$

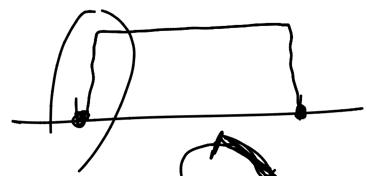
<u>Prop</u>: The kernel of the bounded linear operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ i.e. $H^1_0(\Omega)$ is a closed subspace of $H^1_0(\Omega)$ i.e. Hilbert space with norm $\|\cdot\|_{H^1(\Omega)}$.

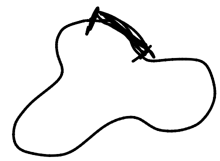
<u>Thm A.6 Poincaré's inequality</u>

If Ω is bounded in $\mathbb{R}^d$

$$\|v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega)$$

The dual space of $H_0^1(\Omega)$ is denoted $H^{-1}(\Omega)$





$$\|L\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|L(v)|}{\|v\|_{H^1(\Omega)}}$$

$$\begin{cases} -\Delta u = f \in H^{-1}(\Omega) \\ u = 0 \end{cases} \quad u \in H_0^1(\Omega)$$

$$|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$$

$$|v|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)}$$

$$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

$$\overset{P}{\le} C^2 \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$$

$$\le (1+C^2)\|\nabla v\|^2_{L^2(\Omega)} \Rightarrow$$

$$\|v\|_{H^1(\Omega)} \le (1+C^2)^{1/2} |v|_{H^1(\Omega)} \quad v \in H_0^1(\Omega)$$

On $H_0^1(\Omega)$, $\|\cdot\|_{H^1}$ and $|\cdot|_{H^1}$ are equivalent.