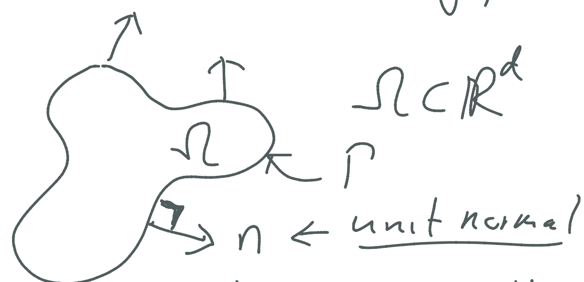
Elliptic partial differential equations

We consider 2nd order PDEs.

$$(*)\begin{cases} Au := -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu = f, & \Omega \\ u = g, & \Gamma \end{cases}$$



$\Omega \subset \mathbb{R}^d$

$a(x) \geq a_0 > 0$, $c(x) \geq 0$, $b(x)$ are smooth.

$a=1, b=0, c=0$ Poisson equation

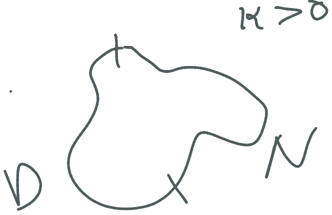
also $f=0$ Laplace equation.

Boundary conditions

1) Dirichlet $u = g$ on Γ
2) Neumann $a\partial_n u := (n\cdot\nabla u)a = g$ on Γ
3) Robin $a\partial_n u + \kappa(u-g) = 0$, $\kappa > 0$

We can combine b.c.

its called mixed b.c.



If $u \in C^2(\bar{\Omega})$ we call it a classical solution.

Thm 3.1 (Maximum principle)

Assume $u \in C^2(\bar{\Omega})$ to (*) and $Au \leq 0$ for all $x \in \Omega$.

* If $c = 0$ then $\max_{x \in \bar{\Omega}} u = \max_{x \in \Gamma} u$

* If $c \geq 0$ then $\max_{x \in \bar{\Omega}} u \leq \max\left(\max_{x \in \Gamma} u, 0\right)$.

If $Au \geq 0$ the same thing holds with minimum.

Proof: $c = 0$,

Proof by contradiction.

At an interior maximum $x \in \Omega$

$\nabla u(x) = 0$

$-\Delta u(x) \geq 0$

$Au(x) = -\nabla \cdot a(x) \nabla u(x) + b(x) \cdot Du(x)$

$= -a(x) \Delta u(x) \geq 0$

almost contradicts the statement.

We perturb u. Let $\phi = e^{\lambda x_1}$

$\mathcal{A}\phi = (-a(x)\lambda^2 + (b_1 - \frac{\partial a}{\partial x_1})\lambda) < 0$ if λ is sufficiently large.

$v = u + \varepsilon\phi$.

For a sufficiently small ε also v will have interior maximum.

$\Rightarrow \nabla v(\xi) = 0 \quad -\Delta v(\xi) \geq 0, \underline{\mathcal{A}v(\xi) \geq 0}$

But $\underline{\mathcal{A}v = \underline{\mathcal{A}u} + \varepsilon \mathcal{A}\phi < 0}$

contradiction

$\blacksquare$

<u>Thm 3.2</u> Assume $u \in C^2(\bar{\Omega})$ solves (*)

then $\|u\|_{C(\bar{\Omega})} \leq \|u\|_{C(\Gamma)} + C\|\mathcal{A}u\|_{C(\bar{\Omega})}$

where $\|v\|_{C(M)} = \sup_{x \in M} |v(x)|$

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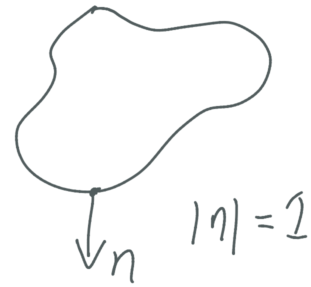
Let u_1 solve (*) with g_1 and f_1

u_2 —"— g_2 and f_2

$$\begin{cases} \mathcal{A}(u_1 - u_2) = f_1 - f_2 \\ (u_1 - u_2) = g_1 - g_2 \end{cases}$$

$\|u_1 - u_2\|_{C(\bar{\Omega})} \leq \|g_1 - g_2\|_{C(\Gamma)} + C\|f_1 - f_2\|_{C(\bar{\Omega})}$

If $g_1 = g_2$, $f_1 = f_2 \Rightarrow u_1 = u_2$ uniqueness.

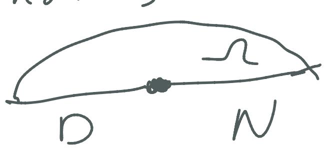


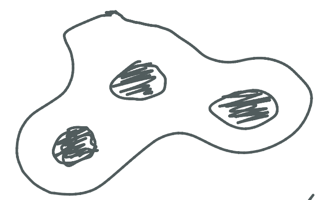
$\partial_n u = n \cdot \nabla u(x)$

Variational formulation

Often $u \in C^2(\bar{\Omega})$ does not hold.

Γ not smooth





$a(x), b(x), c(x)$ may be discontinuous

f may be —"—

g —"—

$$\begin{cases} Au := -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu = f \text{ in } \Omega \\ \underline{\underline{u = 0}} \text{ on } \Gamma \end{cases}$$

We multiply by $v \in C_0^1(\Omega)$ and $\int_\Omega$

$$\int_\Omega f v \, dx = \int_\Omega (-\nabla\cdot) a\nabla u \, v + b\nabla u \, v + c u v \, dx$$

$$= \int_\Omega a\nabla u \cdot \nabla v \, dx + \int_\Omega b\cdot\nabla u \cdot v \, dx + \int_\Omega c u \cdot v \, dx$$

trail test $\forall v \in C_0^1(\Omega)$

Since $C_0^1(\Omega)$ is dense in $H_0^1(\Omega)$

Find a weak solution $u \in H_0^1(\Omega)$ such that

$$\int_\Omega a \nabla u \cdot \nabla v \, dx + \int_\Omega b \cdot \nabla u \cdot v \, dx + \int_\Omega c u v \, dx = \int_\Omega f \cdot v \, dx$$

$\forall v \in H_0^1(\Omega)$.

If u is a classical solution, it is also a weak solution.

If u is a weak solution and $u \in C^2(\bar{\Omega})$ it is also a classical solution.

If $u \in H^2(\Omega) \cap H_0^1(\Omega)$ it is a strong solution.

<u>Thm 3.6</u> Assume $f \in L^2(\Omega)$,

$0 < a_0 \leq a(x) \leq a_1 < \infty$, $|b(x)| \leq b_1$

$c(x) \leq C_1$ $\quad c - \frac{1}{2} \nabla \cdot b \geq 0 \quad \forall x \in \Omega$.

Then there exists a unique solution $u \in H_0^1(\Omega)$ and $\|u\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$

<u>Proof</u>: We equip the Hilbert space $H_0^1(\Omega)$ with the norm $|\cdot|_{H^1(\Omega)} = \|\nabla \cdot\|_{L^2(\Omega)}$

To use Lax-Milgram we need to show $L(v) = \int_\Omega f \cdot v \, dx$ is <u>bounded</u>.

The bilinear form

$$a(u,v) = \int_\Omega a \nabla u \cdot \nabla v \, dx + \int_\Omega b \cdot \nabla u \cdot v dx + \int_\Omega c u \cdot v dx$$

is <u>bounded</u> and <u>coercive</u>. Then there exists unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega).$$

1) $$|L(v)| = \left| \int_\Omega f \cdot v dx \right| \leq \|f\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)}$$

$$\leq C_p \|f\|_{L^2(\Omega)} \cdot \underbrace{|v|_{H^1(\Omega)}}_{\|\nabla v\|_{L^2(\Omega)}}$$

$$\forall v \in H_0^1(\Omega)$$

2) coercivity of a

$$\int_\Omega v \cdot b \cdot \nabla v \, dx = -\int_\Omega v \cdot \nabla \cdot (bv) \, dx =$$

$$= -\int_\Omega v^2 \nabla \cdot b \, dx - \int_\Omega v \, b \cdot \nabla v \, dx$$

$$\int_\Omega v \cdot b \cdot \nabla v \, dx = -\frac{1}{2} \int_\Omega v^2 \nabla \cdot b \, dx$$

$$a(v,v) \geq a_0 |v|^2_{H^1(\Omega)} + \int_\Omega \underbrace{\left(c - \tfrac{1}{2}\nabla\cdot b\right)}_{\geq 0} v^2 \, dx$$

$$\geq a_0 |v|^2_{H^1(\Omega)} .$$

$$|a(u,v)| \leq \left| \int_\Omega a \nabla u \cdot \nabla v \, dx + \int_\Omega b \cdot \nabla u \cdot v + \int_\Omega c u \cdot v \right|$$

$$\overset{C.S.}{\leq} a_1 \|\nabla u\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)} + b_1 \|\nabla u\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)}$$

$$+ c_1 \|u\|_{L^2(\Omega)} \cdot \|v\|_{L^2(\Omega)} \leq$$

$$\leq \left(a_1 + b_1 \underline{C_p + c_1 C_p^2} \right) |u|_{H^1(\Omega)} \cdot |v|_{H^1(\Omega)}$$

$\therefore$ Lax-Milgram guarantees existence and uniqueness of solution.

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + |u|^2_{H^1(\Omega)} \leq$$

$$\leq \left(C_p^2 + 1\right) |u|^2_{H^1(\Omega)} \leq (C_p^2+1) a_0^{-1} \underline{a_0 |u|^2_{H^1(\Omega)}}$$

$$\leq (C_p^2+1) a_0^{-1} a(u,u) = (C_p^2+1) a_0^{-1} L(u)$$

$$\leq \underline{(C_p^2+1) a_0^{-1} C_p} \cdot \|f\|_{L^2(\Omega)} \cdot \cancel{\|u\|_{H^1(\Omega)}}$$

$$\therefore \|u\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)} . \quad \blacksquare$$

Inhomogeneous boundary conditions

$$\begin{cases} Au := -D\cdot aDu + b\cdot Du + cu = f \text{ in } \Omega \\ u = g \text{ on } \Gamma \end{cases}$$

Assume $g = \gamma u_0$, $u_0 \in H^1(\Omega)$

Let $w = u - u_0 \Rightarrow \gamma w = 0$ $w \in H_0^1(\Omega)$

We seek $w \in H_0^1(\Omega)$ s.t.

$$a(w, v) = L(v) - a(u_0, v) \quad \forall v \in H_0^1(\Omega)$$

$$\tilde{L}(v) := L(v) - a(u_0, v)$$

$$|\tilde{L}(v)| \le C_p \|f\|_{L^2(\Omega)} \cdot |v|_{H^1} + M |u_0|_{H^1} \cdot |v|_{H^1}$$

$$\le C |v|_{H^1(\Omega)}$$

$\therefore$ $w \in H_0^1(\Omega)$ exists given u_0.

Let $u_1 = u_0^1 + \underline{w}^1$ $\quad u_2 = u_0^2 + \underline{w}^2$

$$\gamma u_0^1 = \gamma u_0^2 = g$$

$$\gamma(u_1 - u_2) = 0 \Rightarrow u_1 - u_2 \in H_0^1(\Omega)$$

Let $v = u_1 - u_2$

$$a_0 |u_1 - u_2|^2_{H^1(\Omega)} \le a(u_1 - u_2, v) =$$

$$= a(u_1, v) - a(u_2, v) = L(v) - L(v) = 0$$

$\Rightarrow u_1 = u_2$ so unique $\boxed{}$