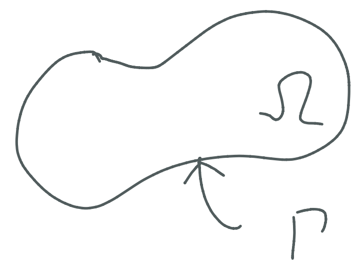
Regularity of solution

$$\begin{cases} -\nabla \cdot a \nabla u = f, & \Omega \\ u = 0, & \Gamma \end{cases}$$



$u \in H^1_0(\Omega)$

Thm If Ω is smooth or convex and $a, b, c \in C^1(\bar{\Omega})$ in addition to original assumptions. Then $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\|v\|_{H^2(\Omega)} \leq C \|Av\|_{L^2(\Omega)} \; \forall v \in H^2 \cap H^1_0$

Here $Au = -\nabla \cdot a \nabla u + b \cdot \nabla u + cu$.

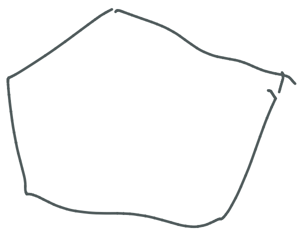
$$a_0 \leq a \leq a_1 \quad \left(c - \tfrac{1}{2} \nabla \cdot b \geq 0\right)$$

$$|c| \leq c_1 \quad |b| \leq b_1$$

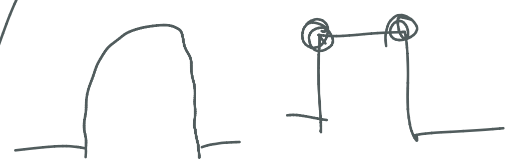
Proof: See Problem 3.11

$$Au = -\Delta u, \quad \Omega = [0,1]^2.$$

$\omega > \pi \quad u \notin H^2(\Omega)$

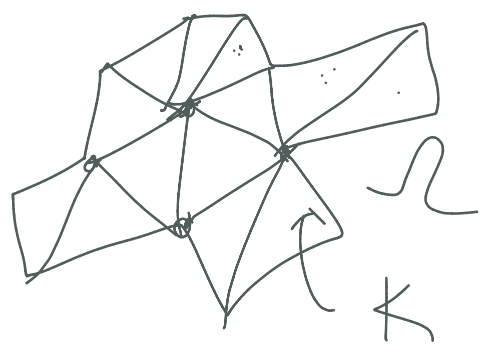


$u \in H^2(\Omega)$

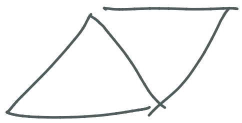


The finite element method

Let $\mathcal{T}_h = \{K\}$ be a triangulation of Ω. $\bigcup_{K \in \mathcal{T}_h} K = \Omega$.

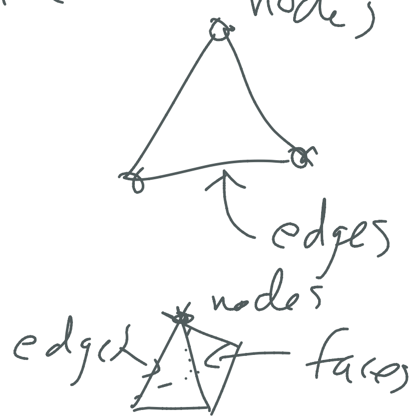


The intersection of two elements is either empty, a node or a common edge.

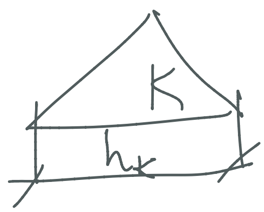


valid

invalid



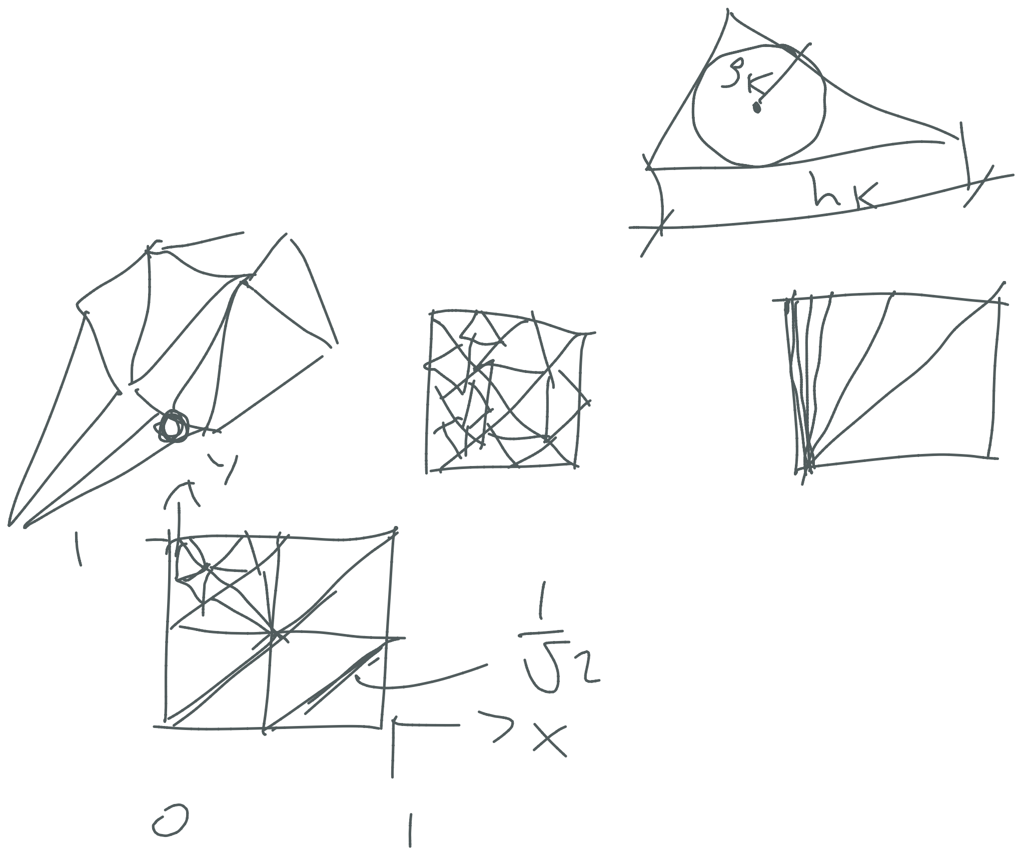
let $h_K = \mathrm{diam}(K)$,



$h = \max_{K \in T_h} h_K$ mesh size.

<u>Def</u>: A family of triangulations $\{T_h\}$ is shape regular if $\kappa > 0$ such that every $K \in T_h$ contains a ball of radius

ρ_K such that $\rho_K \geq \frac{h_K}{\kappa}$.



Let S_h^k be the space of continuous piecewise polynomials of degree k.

$$S_h^k = \left\{ \underline{v \in C(\bar{\Omega})} : v \text{ is polynomial of degree at most } k \text{ on } K,\ K \in \mathcal{T}_h \right\}$$

$k=2$

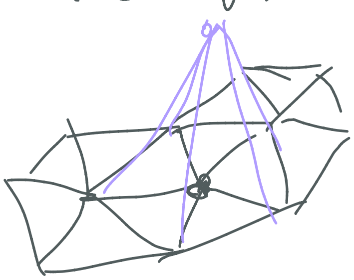
$k=1$

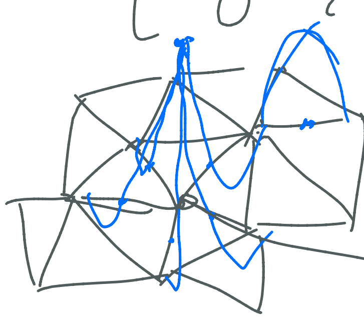
$S_h = S_h^1$.

$V_h = S_h \cap H_0^1(\Omega) = \{v \in S_h : v|_\Gamma = 0\}$

Let $\mathcal{N}$ denote the nodes in T_h and $\{\phi_i\}_{i\in\mathcal{N}} \in S_h$ is the set of tent functions spanning S_h

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$





$\varphi_i = 2\phi_i^2 - \phi_i$

$S_h \subset H^1(\Omega)$, $V_h \subset H^1_0(\Omega)$

A function in S_h is determined by its nodal values. Therefore if $n = |N|$ there is a one-one correspondance between S_h and $\mathbb{R}^n$. Therefore S_h is complete i.e. a closed subset of $H^1(\Omega) \Rightarrow$ Hilbertspace.

Approximation of functions

For $v \in C(\bar{\Omega})$ we define the nodal interpolant

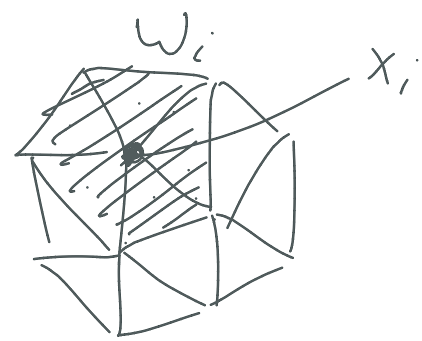
$$I_h v = \sum_{i \in N} v(x_i)\phi_i(x)$$

$$I_h : C(\bar{\Omega}) \to S_h$$

$$\|I_h v - v\|_{L^2(K)} + h_K |I_h v - v|_{H^1(K)} \leq C_K h_K^2 |v|_{H^2(K)}$$

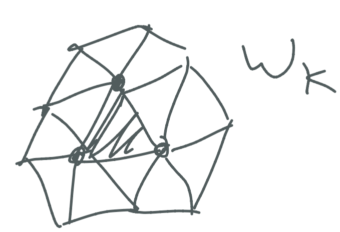
$$\forall K \in T_h, \quad v \in H^2(K).$$

So nodal interpolation is not ideal in rough Sobolev spaces like $H^1(\Omega)$.



$$\omega_i = \cup\{K \in T_h : x_i \in K\}$$

$$\omega_K = \cup\{\omega_i : K \in \omega_i\}$$



$$I_h v = \sum_{i \in N} \frac{\int_{\omega_i} v \, dx}{\int_{\omega_i} 1 \, dx} \phi_i.$$

$I_h : L^1(\Omega) \to S_h$

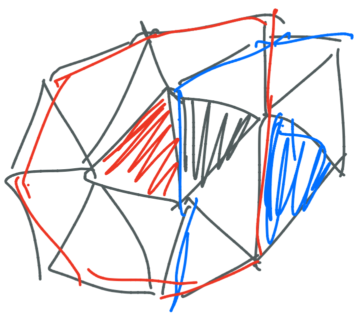
<u>Thm</u>: For any $v \in H^{1+s}(\Omega)$, $s = 0, 1$

$$\|v - I_h v\|_{L^2(K)} + h_K \|v - I_h v\|_{H^1(K)} \leq$$
$$\leq C h_K^{1+s} \|v\|_{H^{1+s}(\omega_K)}$$

$$\|v - I_h v\|_{L^2(\partial K)} \leq C h_K^{1/2+s} \|v\|_{H^{1+s}(\omega_K)}$$

Since T_h is shape regular

$$\|v\|^2_{H^{1+s}(\Omega)} \leq \sum_{K \in T_h} \|v\|^2_{H^{1+s}(\omega_K)} \leq$$
$$\leq C_R \|v\|^2_{H^{1+s}(\Omega)}$$



<u>The finite element method</u>

We consider: find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) = L(v) \quad , \forall v \in H_0^1(\Omega).$$

where $a(u,v) = (a\nabla u, \nabla v) + (b\cdot \nabla u, v) + (cu,v)$

$L(v) = (f,v)$, $\quad u,v \in H_0^1(\Omega)$

with the notation $(v,w) = \int_\Omega v \cdot w \, dx$.

FEM: find $u_h \in V_h \subset H_0^1(\Omega)$

such that $a(u_h, v) = L(v)$, $v \in V_h$.

<u>Thm</u>: $f \in L^2(\Omega)$, $0 < a_0 \leq a \leq a_1 < \infty$

$|b| \leq b_1$, $c \leq c_1$ $\quad c - \frac{1}{2}\nabla\cdot b \geq 0$

$\forall x \in \Omega$. Then there exists

unique solution u_h and

$\|u_h\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$

<u>Proof</u>: Since $V_h \subset H_0^1(\Omega)$ Hilbert

space existence and uniqueness follows by Lax-Milgram since a is bounded and coercive and L is bounded. Furthermore $\|u_h\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$. $\boxtimes$

Implementation

$$u_h = \sum_{j \in N} U_j \phi_j \in V_h$$

find u_h s.t. $\sum_{j \in N} U_j \, a(\phi_j, \phi_i) = L(\phi_i)$ $i \in N$.

Let A be $n \times n$ matrix with

$A_{ij} = a(\phi_j, \phi_i)$. A is sparse.

Let b be $n \times 1$ vector

$$b_i = L(\phi_i) = \int_\Omega f \phi_i \, dx$$

The assembly of A and b

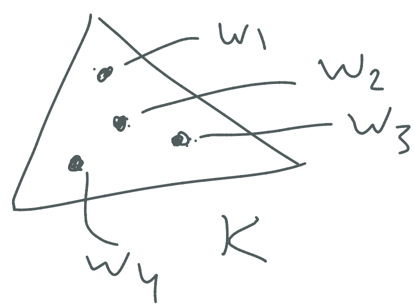
is done over elements K

$$(A^K)_{kl} = (a\nabla\phi_l, \nabla\phi_k)_{L^2(K)} + (b\cdot\nabla\phi_l, \phi_k)_{L^2(K)} + (c\phi_l, \phi_k)_{L^2(K)}$$

$$(b^K)_l = (f, \phi_l)_{L^2(K)}, \quad k, l = 1, \dots, d+1$$

The integrals are computed numerically.

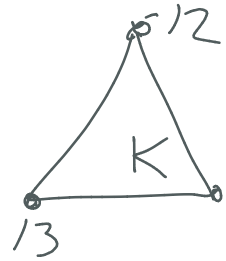
$$\int_K g\,dx \approx \sum_{i=1}^{n} w_i\, g(p_i^K)\,|K|$$



Gauss points are typically used.

Let $n_K \subset N$ be a $d+1$ vector of node numbers of K

Initiate empty A



for $K \in T_h$

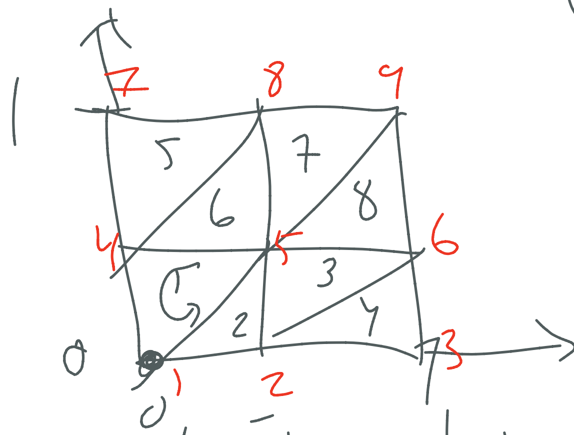
$$A(n_K, n_K) = A(n_K, n_K) + A^K;$$

$$b(n_K) = b(n_K) + b^K;$$

end

$$\text{nodes} = \begin{bmatrix} 0 & 0.1 & \cdots \\ 0 & 0.2 & \cdots \end{bmatrix}$$

$$\text{triangles} = \begin{bmatrix} 13 & 12 & \\ 1 & 17 & \\ 12 & 13 & \cdots \end{bmatrix}$$



$$p = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \end{bmatrix}$$

$$t = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 5 & 5 \\ 5 & 2 & 6 & 3 & 8 & 5 & 9 & 6 \\ 4 & 5 & 5 & 6 & 7 & 8 & 8 & 9 \end{bmatrix}$$

$$e = \begin{bmatrix} \cdots \end{bmatrix}$$