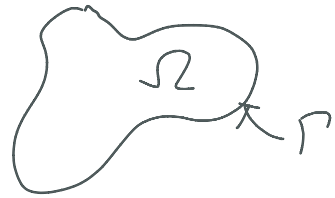
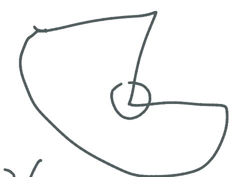
Error analysis

$$\begin{cases} Au := -\nabla\cdot a\nabla u + b\cdot\nabla u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$



FEM: $u_h \in V_h$:

$$a(u_h, v) = (a\nabla u_h, \nabla v) + (b\cdot\nabla u_h, v) + (cu_h, v) = (f, v) \quad \forall v \in V_h.$$



Thm 5.3 & 5.4

$$\|u-u_h\|_{H^1(\Omega)} \le C \min_{\chi\in V_h} \|u-\chi\|_{H^1(\Omega)}$$

$$\|u-u_h\|_{H^1(\Omega)} \le C h \|u\|_{H^2(\Omega)}$$

Assuming $b=c=0$ and

$$\|v\|_{H^2(\Omega)} \le C \|-\nabla\cdot a\nabla u\|_{L^2(\Omega)} \quad v\in H^2\cap H^1_0$$

Elliptic regularity

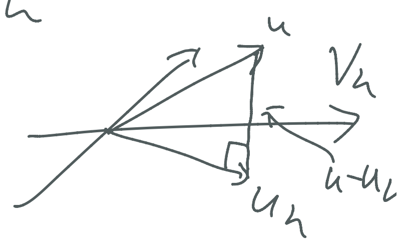
$$\|u-u_h\|_{L^2(\Omega)} \le C h^2 \|u\|_{H^2(\Omega)} \le C h^2 \|f\|_{L^2(\Omega)}$$

__Proof__: Recall $a(u,v) = L(v) \ \forall v \in H_0^1$

$a(u_h, v) = L(v) \ \forall v \in V_h \subset H_0^1$

$$a(u-u_h, v) = 0 \quad \forall v \in V_h$$

Galerkin orthogonality



$$a_0 \|u-u_h\|^2_{H^1(\Omega)} \overset{\text{Coercivity}}{\leq} a(u-u_h, u-u_h) = a(u-u_h, u-\chi)$$

$\chi \in V_h$

$$\overset{\text{Boundedness}}{\leq} M \|u-u_h\|_{H^1(\Omega)} \cdot \|u-\chi\|_{H^1(\Omega)}$$

$$\therefore \ \|u-u_h\|_{H^1(\Omega)} \leq C \|u-\chi\|_{H^1(\Omega)} \quad \forall \chi \in V_h$$

Consider $\chi = I_h u$ Int. bound

$$\|u-u_h\|_{H^1(\Omega)} \leq C \|u - I_h u\|_{H^1(\Omega)} \leq$$

$$\leq C' h \|u\|_{H^2(\Omega)}.$$

$Au = -\nabla \cdot a \nabla u$, $\|v\|_{H^2} \leq \|Av\|_{L^2}$

Aubin-Nitsche trick $v \in H^2 \cap H_0^1$

$$\begin{cases} A\phi = u-u_h \\ \phi = 0 \end{cases}$$ adjoint problem

$$\|u-u_h\|^2_{L^2(\Omega)} = (\mathcal{A}\phi, u-u_h) = (a\nabla\phi, \nabla u-u_h)$$

$$= (a\nabla u-u_h, \nabla\phi) \overset{G.O.}{=} (a\nabla u-u_h, \nabla\phi - I_h\phi)$$

$$\leq M\|u-u_h\|_{H^1(\Omega)} \cdot \|\phi - I_h\phi\|_{H^1(\Omega)}$$

$$\leq Ch\|u\|_{H^2(\Omega)} \cdot Ch\|\phi\|_{H^2(\Omega)}$$

$$\leq Ch^2\|\mathcal{A}u\|_{L^2(\Omega)} \cdot \|\mathcal{A}\phi\|_{L^2(\Omega)}$$

$$\leq Ch^2\|f\|_{L^2(\Omega)} \cdot \|u-u_h\|_{L^2(\Omega)}$$

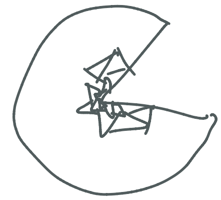
$$\|u-u_h\|_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)} \quad \blacksquare$$

$$\mathcal{A} = -\nabla\cdot a\nabla$$

$$(\mathcal{A}\phi, v) = \int_\Omega -\nabla\cdot a\nabla\phi \cdot v\,dx \overset{\text{Green's}}{=}$$

$$\int_\Omega a\nabla\phi\cdot\nabla v\,dx = a(\phi, v)$$

<u>Thm</u>: Variation of Thm 5.6

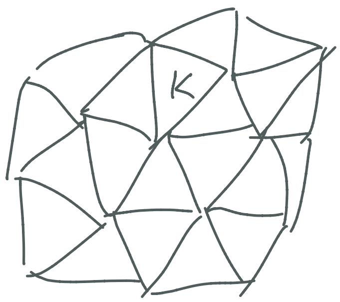


Let $\mathcal{A} = -\nabla\cdot a\nabla + b\cdot\nabla + c$

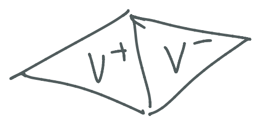
and $a(u_h, v) = l(v)\ \forall v \in V_h$

then $\|u-u_h\|^2_{H^1(\Omega)} \le C \sum_{K\in T_h} R_K^2(u_h)$

where $R_K^2(u_h) = h_K^2 \|A u_h - f\|^2_{L^2(K)} +$



$$+\, h_K \|[n\cdot a\nabla u_h]\|^2_{L^2(\partial K\setminus\Gamma)}$$



$\underbrace{[v]}_{\text{jump}} = v^+ - v^-$

Proof: Let $e = u - u_h$

$$a_0\|e\|^2_{H^1(\Omega)} \underset{\text{coercivity}}{\le} a(e,e) \overset{\text{G.O.}}{=} a(e, e - \overset{\text{Clément}}{I_h} e)$$

$$= \sum_{K\in T_h} \int_K a\nabla e\cdot\nabla(e - I_h e) + b\cdot\nabla e\,(e - I_h e) + $$
$$+\, c\, e\,(e - I_h e)\,dx$$

$$= \sum_{K\in T_h} \int_K (f - c u_h - b\cdot\nabla u_h)(e - I_h e)\,dx \qquad \big(a(u,v) = (f,v)\big)$$
$$- \int_K a\nabla u_h\cdot\nabla(e - I_h e)\,dx$$

$$= \sum_{K\in T_h} \int_K (f - c u_h - b\cdot\nabla u_h + \nabla\cdot a\nabla u_h)(e - I_h e)\,dx$$

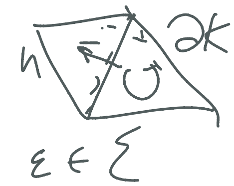
$$-\sum_{K\in\mathcal{T}_h}\int_{\partial K\setminus\Gamma} n\cdot a\nabla u_h\cdot(e-I_h e)\,ds$$

(Green)

$$\left(\int_K a\nabla u_h\nabla v\,dx = \int_K -\nabla\cdot a\nabla u_h v\,dx + \int_{\partial K} n\cdot a\nabla u_h v\,ds\right)$$

$$= \sum_{K\in\tilde{\mathcal{T}}_h}\int_K (f-Au_h)(e-I_h e)\,dx$$

$$-\sum_{\varepsilon\in\mathcal{E}}\int_{\varepsilon\setminus\Gamma}[n\cdot a\nabla u_h](e-I_h e)\,ds$$



$$\le \sum_{K\in\tilde{\mathcal{T}}_h}\|f-Au_h\|_{L^2(K)}\cdot\|e-I_h e\|_{L^2(K)}$$

$$+\left|\sum_{K\in\tilde{\mathcal{T}}_h}\frac{1}{2}\int_{\partial K\setminus\Gamma}[n\cdot a\nabla u_h](e-I_h e)\,ds\right|$$

$$\le \sum_{K\in\tilde{\mathcal{T}}_h} C h_K\|f-Au_h\|_{L^2(K)}\|e\|_{H^1(\omega_K)}$$

$$+\sum_{K\in\tilde{\mathcal{T}}_h} C\,\|[n\cdot a\nabla u_h]\|_{L^2(\partial K\setminus\Gamma)}\cdot\|e-I_h e\|_{L^2(\partial K)}$$

$$\underline{\le \sum_{K\in\tilde{\mathcal{T}}_h}\left(C h_K\|f-Au_h\|_{L^2(K)}+C h_K^{1/2}\|[n\cdot a\nabla u_h]\|_{L^2(\partial K\setminus\Gamma)}\right)}$$

$$\left(\sum_K a_K b_K \le \left(\sum a_K^2\right)^{1/2} \left(\sum b_K^2\right)^{1/2} \right) \cdot \|e\|_{H^1(\omega_K)}$$

$$\le \left(C \sum_{K \in T_h} h_K^2 \|f - Au_h\|^2_{L^2(K)} + h_K \|[n \cdot a \nabla u_h]\|^2_{L^2(\partial K)} \right)^{1/2}$$

$$\cdot \left(\sum_{K \in T_h} \|e\|^2_{H^1(\omega_K)} \right)^{1/2}$$

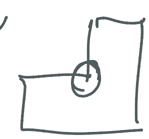
$$\left((a+b)^2 \le 2a^2 + 2b^2 \right)$$

$$\le C \left(\sum_{K \in T_h} R_K^2(u_h) \right)^{1/2} \cdot \|e\|_{H^1(\Omega)}$$

$$\therefore \|e\|^2_{H^1(\Omega)} \le C \sum_{K \in T_h} R_K^2(u_h) \quad \blacksquare$$

Error in a linear functional

$q(u - u_h) \qquad q : V \to \mathbb{R} \qquad q(v) = \int_\omega v \, dx$



We introduce the adjoint problem

$$a(w, \phi) = q(w) \quad , \forall w \in V.$$

<u>Thm</u>: $q(u - u_h) \le \sum_K R_K(u_h) \cdot W_K(\phi)$

where $R_K^2(u_h) = \|Au_h - f\|_{L^2(K)}^2 + h_K^{-1}\|[n\cdot a\nabla u_h]\|_{L^2(\partial K\setminus\Gamma)}^2$

$$W_K^2(\phi) = \|\phi - I\phi\|_{L^2(K)}^2 + h_K\|\phi - I_h\phi\|_{L^2(\partial K\setminus\Gamma)}^2$$

<u>Proof</u>: $q(e) = a(e,\phi) = a(e, \phi - I_h\phi)$

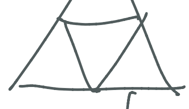
$$= \sum_K \int_K (f - cu_h - b\cdot\nabla u_h)(\phi - I_h\phi)\,dx - \int_K a\nabla u_h \cdot \nabla(\phi - I\phi)\,dx$$

$$\leq \ldots \leq \sum_{K\in T_h} \|f - Au_h\|_{L^2(K)} \cdot \|\phi - I_h\phi\|_{L^2(K)} + h_K^{-1/2}\|[n\cdot a\nabla u_h]\|_{L^2(\partial K\setminus\Gamma)} h_K^{1/2}\|\phi - I\phi\|_{L^2(\partial K\setminus\Gamma)}$$

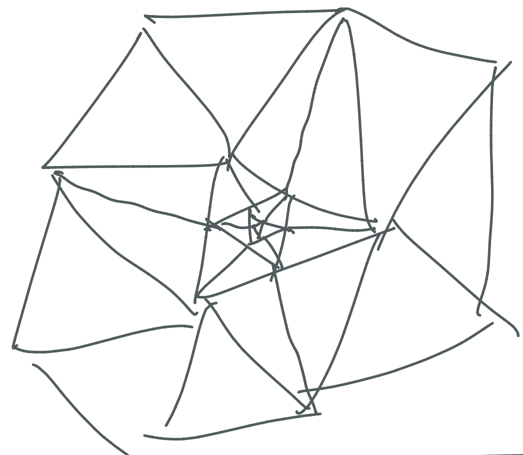
$$\leq C\sum_{K\in T_h} R_K(u_h)\,W_K(\phi).$$

Called goal oriented adaptivity.

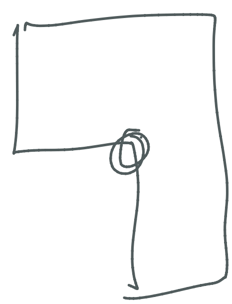
Mesh refinement



No hanging nodes



The aposteriori error bound drives mesh adaptivity



1) Compute U_h
2) Compute $R_K^2(U_h)$
3) Mark element for refinement
4) Refine mesh
5) Stop if error < TOL.