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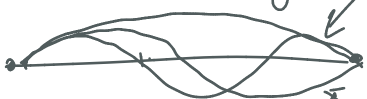
Eigenvalue problems

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \quad u = 0 \end{cases}$$

$\Omega \subset \mathbb{R}^d$

$\Delta = \nabla \cdot \nabla$

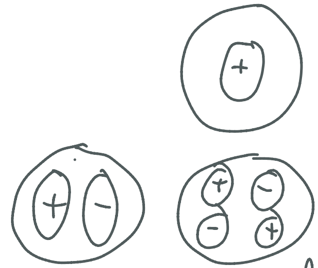
1D Guitar string 110Hz



220Hz

sounds nice

2D Drum



sounds bad

Schrödinger

potential

$$-\Delta u + V u = \lambda u$$

wave function energy

Weak form: find $0 \neq \phi \in V := H_0^1(\Omega)$

and $\lambda \in \mathbb{R}$ such that

$$a(\phi, v) = (\nabla \phi, \nabla v) = \lambda (\phi, v) \quad \forall v \in V$$

We could consider $Au = -\nabla \cdot a \nabla u + \underline{cu}$

Thm 6.1 λ are real and positive.

Two eigenfunctions corresponding to different eigenvalues are orthogonal (in L^2 and H^1

Proof: Let $\{\lambda, \phi\}$ be an eigenpair

Then $\lambda(\phi, \phi) = a(\phi, \phi) \geq a_0 \|\phi\|^2_{H^1(\Omega)}$

$\uparrow$
$v = \phi$

$\lambda \|\phi\|^2_{L^2(\Omega)} \geq a_0 \|\phi\|^2_{H^1(\Omega)} \rightarrow \lambda \in \mathbb{R},\ \lambda > 0$

Let $\{\lambda_1, \phi_1\}$ and $\{\lambda_1, \phi_2\}$ be two eigenpairs

$\lambda_1(\phi_1, \phi_2) = a(\phi_1, \phi_2) = a(\phi_2, \phi_1) = \lambda_2(\phi_2, \phi_1)$

$\uparrow$ $v = \phi_2$ $\qquad$ $\uparrow$ $v = \phi_1$

$(\lambda_1 - \lambda_2)(\phi_1, \phi_2) = 0 \Rightarrow (\phi_1, \phi_2) = 0$

$\Rightarrow a(\phi_1, \phi_2) = 0 \leftarrow H^1$ $\qquad$ $\uparrow L^2$ ∎

Existence of eigenpairs

We say that a set $\mathcal{M} \subset L^2(\Omega)$ is pre-compact if every sequence $\{u_n\}_{n=1}^{\infty} \in \mathcal{M}$ contains a strongly convergent subsequence $\|u_n - u\|_{L^2(\Omega)} \to 0$ as $n \to \infty$

If $\mathcal{M}$ is closed it is compact.

Lemma 6.2 (Rellich)

A bounded set $\mathcal{M}$ of $H^1(\Omega)$ is pre-compact in $L^2(\Omega)$.

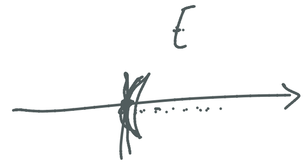
Lemma (Parallelogram Law)

Let $v, w \in V$, where V is a Hilbert space.
Then $\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$

Proof:

$$\|v+w\|^2 + \|v-w\|^2 = (v+w, v+w) + (v-w, v-w)$$
$$= 2\|v\|^2 + 2\|w\|^2 \quad \blacksquare$$

Thm 6.2



The infimum

$$\lambda_1 = \inf\{a(v,v) : v \in H^1_0(\Omega), \|v\|_{L^2(\Omega)} = 1\}$$
$$= \inf_{0 \neq v \in H^1_0(\Omega)} \frac{a(v,v)}{(v,v)}$$

$\leftarrow$ Rayleigh quotient

is attained for a function $\phi_1 \in H^1_0(\Omega)$. ϕ_1 is the eigenfunction corresponding to the eigenvalue λ_1 i.e. $a(\phi_1, v) = \lambda_1(\phi_1, v)$ $\forall v \in V$.

Proof: We take a sequence $\{u_n\}_{n=1}^{\infty} \in H^1_0(\Omega)$ such that $\|\nabla u_n\|^2_{L^2(\Omega)} = a(u_n, u_n) \to \lambda_1$, $\|u_n\| = 1$

This sequence is bounded in $H^1(\Omega)$ therefore there exists a subsequence that is strongly convergent in $L^2(\Omega)$ (Rellich)

We call this subsequence $\{u_n\}_{n=1}^{\infty}$

$u_n \to \phi_1$ in $L^2(\Omega)$

We want to show $\{u_n\}$ converges in $H^1(\Omega)$.

$$\|\nabla(u_n - u_m)\|^2 = 2\|\nabla u_n\|^2 + 2\|\nabla u_m\|^2 - 4\|\tfrac{1}{2}\nabla(u_n+u_m)\|^2$$

$$\leq \left\{ \|\tfrac{1}{2}\nabla(u_n+u_m)\|^2 \geq \lambda_1 \|\tfrac{1}{2}(u_n+u_m)\|^2 \right\}$$

$$\leq 2\|\nabla u_n\|^2 + 2\|\nabla u_m\|^2 - 4\lambda_1 \|\tfrac{1}{2}(u_n+u_m)\|^2$$

$\to \lambda_1$ $\qquad \lambda_1 \qquad \to \phi_1$ and $\|\phi_1\| = 1$

$$\|\nabla(u_n - u_m)\|^2 \to 0 \quad \text{as } n,m \to \infty$$

i.e. a Cauchy sequence in $H_0^1(\Omega)$

$\Rightarrow$ Converges to $\phi_1 \in H_0^1(\Omega)$.

$\lambda_1 = \|\nabla\phi_1\|^2$ with $\|\phi_1\| = 1$.

We want to show $a(\phi_1, v) = \lambda_1(\phi_1, v)$ $\forall v \in H_0^1(\Omega)$

For any $w \in H_0^1(\Omega)$ $\quad \dfrac{a(w,w)}{(w,w)} \geq \lambda_1$

For any $\alpha \in \mathbb{R}$

$$\lambda_1 \leq \frac{a(\phi_1 + \alpha v, \phi_1 + \alpha v)}{(\phi_1 + \alpha v, \phi_1 + \alpha v)} = \frac{\lambda_1 + 2\alpha\, a(\phi_1, v) + \alpha^2 a(v,v)}{1 + 2\alpha(\phi_1, v) + \alpha^2\|v\|^2}$$

$$\lambda_1 + 2\lambda_1\alpha(\phi_1, v) + \lambda_1\alpha^2\|v\|^2 \le \lambda_1 + 2\alpha a(\phi_1, v) + \alpha^2 a(v,v)$$

or $$2\alpha\big(a(\phi_1,v) - \lambda_1(\phi_1,v)\big) + \alpha^2\big(a(v,v) - \lambda_1(v,v)\big) \ge 0$$

If $a(\phi_1, v) \ne \lambda_1(\phi_1, v)$ we can pick α small enough with appropriate sign to violate the inequality. i.e. contradiction i.e

$$a(\phi_1, v) = \lambda_1(\phi_1, v) \quad \forall v \in V. \quad \blacksquare$$

$$\lambda_2 = \inf\{a(v,v) : v \in H_0^1(\Omega), \|v\| = 1, (\phi_1, v) = 0\}$$

$\lambda_2 \ge \lambda_1$ and the inf is attained for $\phi_2 \in H_0^1(\Omega)$ $(\phi_1, \phi_2) = 0$. Furthermore

$$a(\phi_2, v) = \lambda_2(\phi_2, v) \quad \forall v \in \{v \in H_0^1(\Omega) : (v, \phi_1) = 0\}$$

But $a(\phi_1, \phi_2) = \lambda_1(\phi_1, \phi_2) = 0$

and any $v = \alpha\phi_1 + w$ with $w \in \{v \in H_0^1(\Omega) : (w, \phi_1) = 0\}$

$$\Rightarrow a(\phi_2, v) = \lambda_2(\phi_2, v) \quad \forall v \in V$$

Similarly

$$\lambda_n = \inf\{a(v,v) : v \in H_0^1(\Omega), \|v\| = 1, (\phi_j, v) = 0 \;\; j = 1, \dots, n-1\}$$

The number of eigenpairs is infinite.

<u>Thm 6.3</u> It holds $\lambda_n \to \infty$ as $n \to \infty$

Proof: Compactness. Let $\{u_n\}_{n=1}^{\infty}$ orthonormal eigenvectors and assume $\|\nabla u_n\|^2 = \lambda_n \le C$, $n \ge 1$

Then there is a subsequence $\{u_n\}_{n=1}^{\infty}$ that converges in $L^2(\Omega)$ (Rellich)

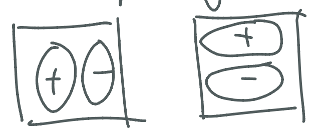
But by orthogonality

$$\|u_n - u_m\|^2 = \|u_n\|^2 + \|u_m\|^2 = 2 \quad \text{contradiction}$$

$\blacksquare$

If $\lambda_n < \lambda_{n+1} = \dots = \lambda_{n+m} < \lambda_{n+m+1}$

we say λ_{n+1} has multiplicity m

Ex: 

Thm 6.4 The eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ form an orthonormal basis of $L^2(\Omega)$ and any $v \in L^2(\Omega)$ $v = \sum_{i=1}^{\infty} (v, \phi_i)\phi_i$

Furthermore $\sum_{j=1}^{\infty} \lambda_j (v, \phi_j)^2$ is convergent iff $v \in H_0^1(\Omega)$

$$\|\nabla v\|^2_{L^2(\Omega)} = a(v,v) = \sum_{j=1}^{\infty} \lambda_j (v, \phi_j)^2$$

Thm 6.5 We have

$$\lambda_n = \min_{V_n \subset H_0^1(\Omega)} \max_{v \in V_n} \frac{a(v,v)}{(v,v)}$$

where V_n varies over all subspaces of $H_0^1(\Omega)$ of dimension n.

<u>Proof</u>: Let $E_n = \text{span}(\{\phi_i\}_{i=1}^n)$

$$\max_{v \in E_n} \frac{a(v,v)}{(v,v)} = \max_{\alpha_1,\dots,\alpha_n} \frac{a(\sum_{i=1}^n \alpha_i \phi_i, \sum_{j=1}^n \alpha_j \phi_j)}{(\sum_{i=1}^n \alpha_i \phi_i, \sum_{j=1}^n \alpha_j \phi_j)}$$

$$= \max_{\alpha_1,\dots,\alpha_n} \frac{\sum_{j=1}^n \alpha_j^2 \lambda_j}{\sum_{j=1}^n \alpha_j^2} = \lambda_n$$

It remains to show for any V_n that $\max_{v \in V_n} \frac{a(v,v)}{(v,v)} \geq \lambda_n$. We pick $\overset{w \neq 0}{w} \in V_n$ such that $(w, \phi_j) = 0 \quad \forall j = 1, \dots, n-1$

Otherwise V_n can not have dim n all members of V_n would be linear combinations of $\{\phi_i\}_{i=1}^{n-1}$.

Since $\lambda_n = \min_{\substack{(v,\phi_i)=0, \\ i=1,\dots,n-1}} \frac{a(v,v)}{(v,v)} \leq \frac{a(w,w)}{(w,w)}$

$$\leq \max_{v \in V_n} \frac{a(v,v)}{(v,v)} \quad \blacksquare$$

<u>Lemma 6.1</u> Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis in $L^2(\Omega)$. Then the best approximation of $v \in L^2(\Omega)$ by a linear combination of

$\{\phi_i\}_{i=1}^N$ is $v_N = \sum_{i=1}^N (v,\phi_i)\phi_i$.

<u>Proof</u>: $\| v - \sum_{i=1}^n \alpha_i \phi_i \|^2 = \|v\|^2 - 2\sum_{i=1}^N \alpha_i (v,\phi_i) + \sum_{i=1}^N \alpha_i^2$

$$= \|v\|^2 + \sum_{i=1}^N (\alpha_i - (v,\phi_i))^2 - \sum_{i=1}^N (v,\phi_i)^2$$

which is minimal when $\alpha_i = (v,\phi_i)$, $i=1,\dots,N$ ∎

Parseval $\sum_{i=1}^\infty (v,\phi_i)^2 = \|v\|^2$

If $a(u,v) = (a\nabla u,\nabla v) + (cu,v) = \lambda(u,v)$

and $0 < a_0 \le a \le a_1$, $0 \le c \le c_1$

$$\lambda_1 = \min_{v\in H_0^1(\Omega)} \frac{(a\nabla v,\nabla v)+(cv,v)}{(v,v)}$$