Finite element method for eigenvalue problems

Find $u \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}^+$ such that

$$a(u,v) = (\nabla u, \nabla v) = \lambda (u,v) \quad \forall v \in H_0^1(\Omega)$$

$$0 < \lambda_1 < \lambda_2 \leq \dots$$

Find $U \in V_h$ and $\Lambda \in \mathbb{R}^+$

$$a(U,v) = (\nabla U, \nabla v) = \Lambda (U,v) \quad \forall v \in V_h$$

$\dim(V_h) = N$. We denote the discrete eigenfunctions $\{\Phi_i\}_{i=1}^N$ corresponding to $\{\Lambda_i\}_{i=1}^N$ real and positive.

__Cor__: We have $1 \leq n \leq N$

$$\Lambda_n = \min_{W_n \subset V_h} \max_{v \in W_n} \frac{a(v,v)}{(v,v)}$$

where n varies over n dim subspaces of V_h.

Error in eigenvalue

Def: Let $R_h : H_0^1(\Omega) \to V_h$ defined by

$$a(R_h v, w) = a(v, w) \quad \forall w \in V_h$$

R_h is called the Ritz projection.

Recall:
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall v \in V_h$$

$$\Rightarrow (\nabla \underset{\substack{\uparrow \\ \in V_h}}{u_h}, \nabla v) = (\nabla u, \nabla v) \quad \forall v \in V_h$$

$$\therefore \quad u_h = R_h u$$

Using Thm 5.4 we get

$$\|v - R_h v\|_{L^2(\Omega)} \le C h^2 \|\Delta v\|_{L^2(\Omega)}$$

if Ω is convex

for any $v \in H^2(\Omega) \cap H^1_0(\Omega)$

Since $\|D^2 v\|_{L^2(\Omega)} \le C \|\Delta v\|_{L^2(\Omega)}$.

$$\|v - R_h v\|_{L^2(\Omega)} \le C h \|v\|_{H^1(\Omega)}$$

But it only works if $v \in H^2(\Omega) \cap H^1_0(\Omega)$

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Recall: $a(v, \phi) = (v, u - u_h)$

$\forall v \in H^1_0(\Omega)$

$$\|u - u_h\|^2 = a(u - u_h, \phi) \le$$

$$\overset{C.S.}{\le} \|\nabla u - u_h\| \cdot \|\nabla \phi\| \le$$

$$\le \left\{ \begin{array}{l} v = \phi \quad \|\nabla \phi\|^2 = (\phi, u - u_h) \le \|\phi\| \cdot \|u - u_h\| \\ \le C \cancel{\|\nabla \phi\|} \cdot \|u - u_h\| \end{array} \right\}$$

$$\le C \|\nabla u - u_h\| \cdot \|u - \cancel{u_h}\|$$

$$\le \|u - u_h\| \le C \|\nabla u - u_h\| \le C h \|\Delta u\|$$

or

$$\|u-u_h\|^2 = a(u-u_h,\phi) = a(u-u_h, \phi - I_h\phi)$$

$$\le \|\nabla(u-u_h)\| \cdot \|\nabla(\phi - I_h\phi)\| \le \left\{ \begin{array}{l} \|\nabla(v - I_h v)\| \le \\ \le Ch \|D^2 v\| = \\ \le C'h\|\Delta v\| \end{array} \right\}$$

$$\le \left(\|\nabla u\| + \|\nabla u_h\|\right) \cdot C\underline{h}\|\Delta\phi\|$$

$$\le \left(\|\nabla u\| + \|\nabla R_h u\|\right) Ch \|u-u_h\|$$

$$\le C' \|\nabla u\| \cdot h \cdot \|u-u_h\|$$

$$\therefore \|u-u_h\| \le Ch\|\nabla u\| .$$

$$\|\nabla R_h v\| \le \|\nabla v\| \quad \forall v \in H^1_0(\Omega)$$

$$(\nabla R_h v, \nabla w) = (\nabla v, \nabla w) \quad w \in V_h$$

$$\|\nabla R_h v\|^2 = (\nabla v, \nabla R_h v) \le \|\nabla v\| \cdot \cancel{\|\nabla R_h v\|}$$

<u>R_h is stable in $H^1_0(\Omega)$</u>.

<u>Thm 6.7</u>

$$\left(-\Delta u_i = \lambda_i u_i\right) \quad \|u_i\| = 1$$

Let Ω be convex (so any $v \in E_n = \operatorname{span}\{u_i\}_{i=1}^n$ fulfills $v \in H^2(\Omega) \cap H^1_0(\Omega)$ and

$\|D^2 v\| \le C\|\Delta v\|$.) Then

$$\lambda_n \le \Lambda_n \le \lambda_n + Ch^2, \quad \forall h \le h_0$$

Proof: By min-max

$$\lambda_n = \min_{V_n \subset H_0^1(\Omega)} \max_{v \in V_n} \frac{a(v,v)}{(v,v)}$$

$$\le \min_{V_n \subset V_h} \max_{v \in V_n} \frac{a(v,v)}{(v,v)} = \Lambda_n$$

Let $E_n = \mathrm{span}(\{u_i\}_{i=1}^n)$ and let

$E_n^h = \mathrm{span}(\{U_i\}_{i=1}^n)$. It holds

$$\max_{v \in E_n} \frac{a(v,v)}{(v,v)} = \max_{\alpha_1,\dots,\alpha_n} \frac{\sum_{i=1}^n \alpha_i^2 \lambda_i}{\sum_{i=1}^n \alpha_i^2} = \lambda_n$$

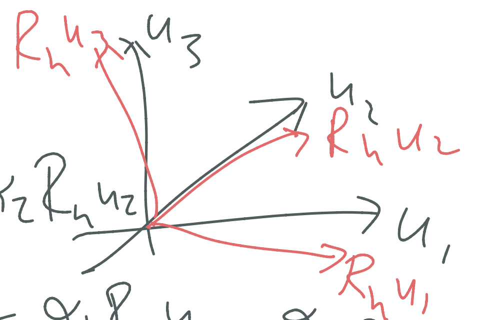
$$\max_{v \in E_n^h} \frac{a(v,v)}{(v,v)} = \Lambda_n$$

We note that $R_h E_n := \{R_h v : v \in E_n\} \subset V_h$

is $\dim(R_h E_n) = n$.

If not

$$R_h u_3 = \alpha_1 R_h u_1 + \alpha_2 R_h u_2$$



$$\underbrace{u_3 - R_h u_3}_{\le C h^2 \|\Delta u_3\|} = u_3 - \alpha_1 R_h u_1 - \alpha_2 R_h u_2 \pm \alpha_1 u_1 \pm \alpha_2 u_2 \ge 1 - C h^2 .$$

We have

$$\Lambda_n = \max_{v \in E_n^h} \frac{a(v,v)}{(v,v)} = \min_{W_n \subset V_h} \max_{v \in W_n} \frac{a(v,v)}{(v,v)}$$

$$\le \max_{v \in R_h E_n} \frac{a(v,v)}{(v,v)} = \max_{v \in E_n} \frac{a(R_h v, R_h v)}{(R_h v, R_h v)}$$

$a(R_h v, R_h v) = a(v, R_h v) \le \|\nabla v\| \cdot \|\nabla R_h v\|$

$$\le \max_{v \in E_n} \frac{a(v,v)}{(R_h v, R_h v)} \le$$

$$(R_h v, R_h v) = \|R_h v\|^2$$

$$\le \max_{v \in E_n} \frac{a(v,v)}{\left(\|v\| - \|v - R_h v\|\right)^2} \quad \text{holds}$$

if $\|v - R_h v\| \le C h^2 \|\Delta v\| \le \underline{C h^2 \lambda_n} \|v\|$

i.e. $\underline{C h^2 \lambda_n} < \frac{1}{2}$.

$$\Lambda_n \le \frac{1}{(1 - C\lambda_n h^2)^2} \underbrace{\max_{v \in E_n} \frac{a(v,v)}{(v,v)}}_{\lambda_n}$$

$$\le \frac{\lambda_n}{(1 - C\lambda_n h^2)^2} \le \lambda_n \left(1 + C' \lambda_n h^2\right)$$

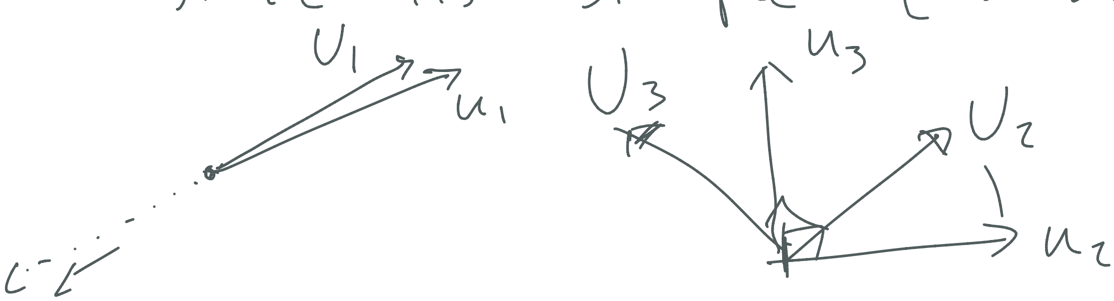
for $h < h_0$ $\left(C h_0^2 \lambda_n < \frac{1}{2} \right)$

$$\lambda_n \le \Lambda_n \le \lambda_n + C h^2 \qquad \blacksquare$$

$$\begin{cases} -\Delta u = f \\ u = 0 \end{cases} \qquad \begin{cases} -\Delta u = \lambda u \\ u = 0 \end{cases}$$

<u>Error in eigenfunction</u>

We only consider the eigenfunction since its simple (not multiple)



Thm 6.8 Let u_1 be the first eigenfunction and U_1 its FE approx. $\|u_1\| = \|U_1\| = 1$. Furthermore $u_1 \in H^2 \cap H^1_0$

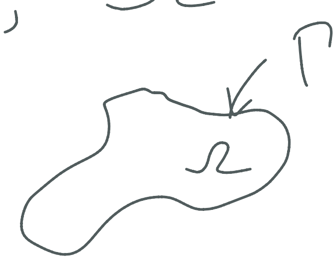
Then $\|u_1 - U_1\| \leq Ch^2$

$$\|u_1 - U_1\|_{H^1} \leq Ch.$$

Parabolic problems

Eigenfunction expansion.

$$\dot{u} - \Delta u = 0 \quad , \quad \Omega \times \mathbb{R}^+$$
$$u = 0 \quad , \quad \Gamma \times \mathbb{R}^+$$
$$u(\cdot, 0) = v \quad , \quad \Omega$$



Let $\{\phi_i\}_{i=1}^{\infty}$ be ON basis of eigenfunctions

find $\phi_i \in H^1_0(\Omega)$: $a(\phi_i, w) = (\nabla\phi_i, \nabla w) = \lambda_i(\phi_i, w)$ $\forall w \in H^1_0(\Omega)$

$0 < \lambda_1 < \lambda_2 \leq \dots \to \infty$ as $n \to \infty$

We seek a solution $u(x,t) = \sum_{i=1}^{\infty} \hat{u}_i(t) \phi_i(x)$

We plug this into the equation

$$0 = \dot{u} - \Delta u = \sum_{i=1}^{\infty} \dot{\hat{u}}_i(t) \phi_i(x) - \hat{u}_i(t) \Delta \phi_i(x)$$

$$= \sum_{i=1}^{\infty} \left(\dot{\hat{u}}_i(t) + \lambda_i \hat{u}_i(t) \right) \underline{\phi_i(x)} \Rightarrow$$

$$\dot{\hat{u}}_i(t) + \lambda_i \hat{u}_i(t) = 0 \Rightarrow$$

$\hat{u}_i(t) = \hat{u}_i(0) e^{-\lambda_i t}$. Moreover

$$u(\cdot, 0) = \sum_{i=1}^{\infty} \hat{u}_i(0) \phi_i(x) = v = \sum_{i=1}^{\infty} (v, \phi_i) \phi_i$$

$$\Rightarrow \hat{u}_i(0) = (v, \phi_i)$$

$$\therefore \quad u(x,t) = \sum_{i=1}^{\infty} (v, \phi_i) e^{-\lambda_i t} \phi_i(x)$$

$$\|u(\cdot, t)\|^2 = \sum_{i=1}^{\infty} (v, \phi_i)^2 e^{-2\lambda_i t} \leq$$

$$\leq e^{-2\lambda_1 t} \sum_{i=1}^{\infty} (v, \phi_i)^2 = e^{-2\lambda_1 t} \|v\|^2$$

$\therefore \|u(\cdot,t)\| \le e^{-\lambda_1 t}\|v\|$

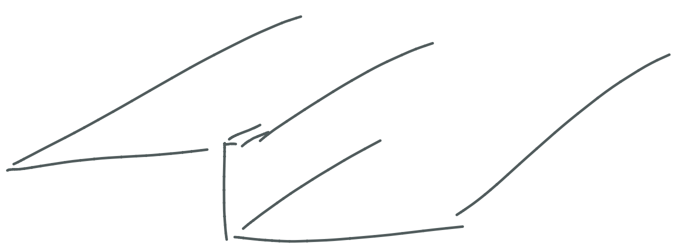
<u>Thm 8.3</u> For any $v \in L^2(\Omega)$ and bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary, the function

$$u(x,t) = \sum_{i=1}^{\infty} (v,\phi_i)\, e^{-\lambda_i t}\, \phi_i(x)$$

is a classical solution to $\begin{cases} u_t - \Delta u = 0 \\ u = 0 \\ u(\cdot,0) = v \end{cases}$

for $t>0$. Moreover it is smooth for $t>0$.

(Parabolic smoothing)



hyperbolic

parabolic