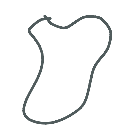
Parabolic problems

$$
(*)\quad \begin{cases} \dot{u} - \Delta u = 0 & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \Gamma \times (0,T] \\ u(\cdot,0) = v & \text{in } \Omega \end{cases}
$$

Let $\begin{cases} -\Delta \phi_i = \lambda_i \phi_i, & \text{in } \Omega \\ \phi_i = 0, & \text{on } \Gamma \end{cases}$

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \to \infty$$

$$u(x,t) = \sum_{i=1}^{\infty} (v,\phi_i) e^{-\lambda_i t} \phi_i(x)$$

Thm 8.3 For any $v \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^d$ with smooth boundary  the function $u = \sum_{i=1}^{\infty} (v,\phi_i) e^{-\lambda_i t} \phi_i(x)$ is a classical solution to $*$. Furthermore it is smooth $t > 0$.

Proof: We note that $s^k e^{-s} \le C_k \ \forall s \ge 0$.

$$\|u(\cdot,t)\|^2_{H^1(\Omega)} \overset{\text{Thm 6.4}}{=} \sum_{i=1}^{\infty} \lambda_i (v,\phi_i)^2 e^{-2\lambda_i t} =$$

$$= \frac{1}{2t} \sum_{i=1}^{\infty} \underbrace{2t\lambda_i e^{-2t\lambda_i}}_{\leq \; k=1} (v,\phi_i)^2 \leq$$

$$\leq \frac{C_1}{2t} \|v\|^2_{L^2(\Omega)} \implies$$

$$|u(\cdot,t)|_{H^1(\Omega)} \leq C t^{-1/2} \|v\|_{L^2(\Omega)}$$

Using Thm 6.4 $\implies u(\cdot,t) \in H^1_0(\Omega)$, $t>0$

Since $\|u(\cdot,t)\| \leq \|v\|$ $\quad \|u(\cdot,t)\|_{H^1(\Omega)} \leq$

$$\leq C t^{-1/2} \|v\|.$$

Apply $(-\Delta)^k$

$$(-\Delta)^k u = \sum_{i=1}^{\infty} (v,\phi_i) \lambda_i^k e^{-\lambda_i t} \phi_i(x)$$

$$\|(-\Delta)^k u\|^2 = \sum_{i=1}^{\infty} (v,\phi_i)^2 \lambda_i^{2k} e^{-2\lambda_i t}$$

$$\leq C_k^2 \frac{1}{t^{2k}} \sum_{i=1}^{\infty} (v,\phi_i)^2 \leq C_k^2 t^{-2k} \|v\|^2$$

$k=1$ $\quad \|\Delta u\| \leq C t^{-1} \|v\|$

Elliptic regularity $\|D^2 u\| \leq C\|\Delta u\|$

$$\|u(\cdot,t)\|_{H^s(\Omega)} \leq C t^{-s/2} \|v\|_{L^2(\Omega)}$$

Repeating the argument

$$\|u(\cdot,t)\|_{H^s(\Omega)} \leq C\, t^{-s/2} \|v\|_{L^2(\Omega)} \quad s \geq 0.$$

Since $D_t^m e^{-\lambda_i t} = (-\lambda_i)^m e^{-\lambda_i t}$ so

$$\|D_t^m \Delta^k u\|^2_{L^2(\Omega)} = \sum_{i=1}^{\infty} (v, \phi_i)^2 \lambda_i^{2(k+m)} e^{-2\lambda_i t}$$

$$\leq C_{k+m}^2\, t^{-2(k+m)} \|v\|^2$$

$$\|D_t^m u(\cdot,t)\|_{H^s(\Omega)} \leq C\, t^{-m-\frac{s}{2}} \|v\|_{L^2(\Omega)}$$

Sobolev inequality $D_t^m u \in C^p(\Omega)$ for $t>0$ any $p \geq 0$.

Therefore u is smooth for $t > 0$.
u also fulfills boundary conditions since the trace of ϕ_i are zero.

$$\|u(\cdot,t) - v\|^2_{L^2(\Omega)} = \sum_{i=1}^{\infty} (e^{-\lambda_i t} - 1)^2 (v, \phi_i)^2$$

Since $v \in L^2(\Omega) \Rightarrow \sum_{i=1}^{\infty} (v, \phi_i)^2 < \infty$

There is an N s.t. $\sum_{i=N+1}^{\infty} (v, \phi_i)^2 < \varepsilon$

$$\sum_{i=1}^{N} (e^{-\lambda_i t} - 1)^2 (v, \phi_i)^2 \to 0 \text{ as } t \to 0$$

For sufficiently small t we get

$$\|u(\cdot, t) - v\|_{L^2(\Omega)} \leq 2\varepsilon$$

We conclude $u(\cdot, 0) = v$ ∎

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Thm 8.4 $v \in H_0^1(\Omega)$ and Γ is smooth

Then $|u(\cdot, t)|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)}$.

Proof: Thm 6.4

$$|u(\cdot, t)|^2_{H^1(\Omega)} = \sum_{i=1}^{\infty} \lambda_i (v, \phi_i)^2 e^{-2\lambda_i t}$$

$$\leq |v|^2_{H^1(\Omega)}$$ ∎

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Inhomogeneous equation

$$(**) \quad \begin{cases} \dot{u} - \Delta u = f, & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0, & \text{on } \Gamma \times \mathbb{R}^+ \end{cases}$$

$$\left\{ \quad u(\cdot,0) = v \;, \; \text{in } \Omega . \right.$$

Assume $f(t) = \sum_{i=1}^{\infty} \beta_i(t)\,\phi_i(x)$ and let $u = \sum_{i=1}^{\infty} \alpha_i(t)\,\phi_i(x)$. We plug this into $**$

$$\sum_{i=1}^{\infty} \dot\alpha_i(t)\,\phi_i + \lambda_i \alpha_i(t)\,\phi_i = \sum_{i=1}^{\infty} \beta_i(t)\,\phi_i$$

$$\dot\alpha_i(t) + \lambda_i \alpha_i(t) = \beta_i(t) \quad \forall i$$

$$D_t\left(\alpha_i(t)\,e^{\lambda_i t}\right) = \dot\alpha_i(t)\,e^{\lambda_i t} + \alpha_i(t)\,\lambda_i\,e^{\lambda_i t}$$

$$= e^{\lambda_i t}\,\beta_i(t) \quad \Rightarrow$$

$$\alpha_i(t)\,e^{\lambda_i t} - \alpha_i(0) = \int_0^t e^{\lambda_i s}\,\beta_i(s)\,ds \Rightarrow$$

$$\Rightarrow \alpha_i(t) = \underbrace{\alpha_i(0)}_{(v,\phi_i)}\,e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-s)}\,\underbrace{\beta_i(s)}_{(f(s),\,\phi_i)}\,ds$$

We conclude

$$u(x,t) = \sum_{i=1}^{\infty}\left((v,\phi_i)\,e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-s)}\,(f(s),\phi_i)\,ds\right)\phi_i$$

We get

$$\| u(t)\|_{L^2(\Omega)} \le \Big\| \sum_{i=1}^{\infty} (v, \phi_i) e^{-\lambda_i t} \phi_i \Big\|_{L^2} +$$

$$\Big\| \sum_{i=1}^{\infty} \int_0^t e^{-\lambda_i (t-s)} (f(s), \phi_i) ds\, \phi_i \Big\|_{L^2(\Omega)}$$

$$\le \Big(\sum_{i=1}^{\infty} (v, \phi_i)^2 e^{-2\lambda_i t} \Big)^{1/2} +$$

$$+ \Big\| \int_0^t \sum_{i=1}^{\infty} e^{-\lambda_i (t-s)} (f(s), \phi_i)\, \phi_i \, ds \Big\|_{L^2(\Omega)}$$

$$\le \|v\|_{L^2(\Omega)} + \int_0^t \Big\| \sum_{i=1}^{\infty} e^{-\lambda_i (t-s)} (f(s), \phi_i)\, \phi_i \Big\|_{L^2(\Omega)} ds$$

$$\le \|v\|_{L^2(\Omega)} + \int_0^t \Big(\sum_{i=1}^{\infty} \underbrace{e^{-2\lambda_i (t-s)}}_{\le 1} (f(s), \phi_i)^2 \Big)^{1/2} ds$$

$$\le \|v\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} ds$$

If $\begin{cases} \dot u_1 - \Delta u_1 = f_1 \\ u_1(0) = v_1 \end{cases}$ $\begin{cases} \dot u_2 - \Delta u_2 = f_2 \\ u_2(0) = v_2 \end{cases}$

$$\|u_1 - u_2\|_{L^2(\Omega)} \le \|v_1 - v_2\|_{L^2(\Omega)} + \int_0^t \|f_1(s) - f_2(s)\| ds$$

Uniqueness of solution.

Variational formulation

Find $u \in H^1_0(\Omega)$ for each $t > 0$ such that

$$(\dot{u}, w) + a(u, w) = (f, w) \quad \forall w \in H^1_0(\Omega)$$

$$a(u, w) = \int_\Omega \nabla u \cdot \nabla w \, dx \quad . \quad u(\cdot, 0) = v \text{ in } \Omega$$

If u is sufficiently smooth we can use Green's formula

$$(\dot{u} - \Delta u - f, w) = 0 \quad \forall w \in H^1_0(\Omega)$$

will only hold if $\dot{u} - \Delta u = f$.

Thm 8.5 Assume u is a weak solution

(i) $$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t |u(s)|^2_{H^1(\Omega)} ds \leq \|v\|^2 + C\int_0^t \|f(s)\|^2 ds$$

(ii) $$|u(t)|^2_{H^1(\Omega)} + \int_0^t \|\dot{u}(s)\|^2_{L^2(\Omega)} ds \leq |v|^2_{H^1(\Omega)} + \int_0^t \|f(s)\|^2 ds$$

Proof: (i) Let $w = u$

Note that $(\dot{u}, u) = \int_\Omega \dot{u} \cdot u \, dx = \int_\Omega \frac{1}{2} \frac{d}{dt} u^2 dx$

$$= \frac{1}{2} \frac{\partial}{\partial t} \|u(t)\|^2 = \|u(t)\| \frac{\partial}{\partial t} \|u(t)\|$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|u(t)\|^2 + \underline{|u(t)|^2_{H^1(\Omega)}} = (\dot{u},u) + a(u,u)$$

$$= (f,u) \overset{C.S.}{\leq} \|f\|\cdot\|u\| \overset{P.F.}{\leq} \underbrace{C\|f\|}_{a}\cdot\underbrace{|u|_{H^1(\Omega)}}_{b}$$

$$\leq \left\{\begin{array}{l} ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \\ (a-b)^2 \geq 0 \\ a^2 - 2ab + b^2 \geq 0 \end{array}\right\} \leq \frac{1}{2}C^2\|f\|^2 + \underline{\frac{1}{2}|u|^2_{H^1(\Omega)}}$$

$$\frac{\partial}{\partial t}\|u(t)\|^2 + |u(t)|^2_{H^1(\Omega)} \leq C\|f\|^2$$

Integrate $\int_0^t$

$$\|u(t)\|^2 - \|v\|^2 + \int_0^t |u(s)|^2_{H^1(\Omega)}\,ds \leq C\int_0^t \|f(s)\|^2 ds$$

(ii) $w = \dot{u}$

$$a(u,\dot{u}) = \int_\Omega \nabla u\cdot\nabla\dot{u}\,dx = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \nabla u\cdot\nabla u\,dx$$

$$= \frac{1}{2}\frac{\partial}{\partial t}|u|^2_{H^1(\Omega)}$$

$$\underline{\|\dot{u}\|^2_{L^2(\Omega)}} + \frac{1}{2}\frac{\partial}{\partial t}|u|^2_{H^1(\Omega)} = (\dot{u},\dot{u}) + a(u,\dot{u}) =$$

$$= (f,\dot{u}) \overset{C.S.}{\leq} \|f\|\cdot\|\dot{u}\| \leq \frac{1}{2}\|f\|^2 + \underline{\frac{1}{2}\|\dot{u}\|^2}$$

$$\|\dot{u}\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}|u|^2_{H^1(\Omega)} \leq \|f\|^2$$

Integrate $\int_0^t$

$$\int_0^t \|\dot{u}(s)\|^2 ds + |u(t)|^2_{H^1(\Omega)} - \underline{|v|^2_{H^1(\Omega)}} \le \int_0^t \|f(s)\|^2 ds$$

$\blacksquare$

Again $\begin{cases} \dot{u}_1 - \Delta u_1 = f_1 \\ u_1(0) = v_1 \end{cases}$ $\begin{cases} \dot{u}_2 - \Delta u_2 = f_2 \\ u_2(0) = v_2 \end{cases}$

$$\|u_1(t) - u_2(t)\|^2 + \int_0^t |u_1(s) - u_2(s)|^2_{H^1(\Omega)} \le \|v_1 - v_2\|^2 + C\int_0^t \|f_1(s) - f_2(s)\|^2 ds.$$