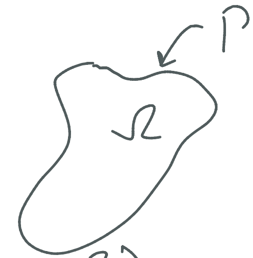
Parabolic maximum principle

$$\dot u - \Delta u = f, \quad \Omega \times I$$

$$I = (0,T)$$

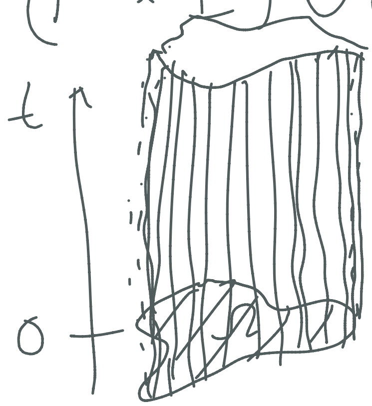
$$u = g, \quad \Gamma \times I$$

$$u(\cdot, 0) = v, \quad \Omega$$



The parabolic boundary

$$\Gamma_p = (\Gamma \times I) \cup (\Omega \times \{t=0\})$$



$$\partial(\Omega \times I) \setminus \{\Omega \times \{t=T\}\}$$

Thm 8.6 Let u be smooth and assume $\dot u - \Delta u \le 0$ in $\Omega \times I$. Then u attains its maximum on the parabolic boundary Γ_p.

Proof:
Assume the statement is not true. Then we either have an interior

maximum in $\Omega \times I$ or on the top $\Omega \times \{t = T\}$. Then there would be a point $(\bar{x}, \bar{t}) \in \Omega \times (0, T]$ such that $u(\bar{x}, \bar{t}) = \max\limits_{\bar{\Omega} \times \bar{I}} u = M > m = \max\limits_{\Gamma_p} u$

Let $w(x,t) = u(x,t) + \varepsilon |x|^2$

$\uparrow x_1^2 + x_2^2 + \ldots + x_d^2$

then if ε is sufficiently small also w will attain its maximum in $\Omega \times (0, T]$

$$\max_{\Gamma_p} w \leq m + \varepsilon \max_{\Gamma_p} |x|^2 < M \leq \max_{\bar{\Omega} \times \bar{I}} w$$

$$\Delta(|x|^2) = 2d$$

$$\dot{w} - \Delta w = \dot{u} - \Delta u - 2d\varepsilon < 0$$

However if w attains its maximum in $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T]$ then

$$-\Delta w(\tilde{x}, \tilde{t}) \geq 0$$

and $\dot{w}(\tilde{x}, \tilde{t}) = 0$, $0 < \tilde{t} < T$

or $\dot{w}(\tilde{x}, T) \geq 0$

so $(\dot{w} - \Delta w)(\tilde{x}, \tilde{t}) \geq 0$ Contradiction $\blacksquare$

If $f=0$ and we consider $\pm u$

(where $\begin{cases} \dot{u} - \Delta u = 0 \\ u = g \\ u(\cdot,0) = v \end{cases}$)

then it follows that both max and min occurs on Γ_p

$$\max_{\bar{\Omega}\times\bar{I}} |u| = \max\left(\max_{\Gamma\times\bar{I}} |g|, \max_{\bar{\Omega}} |v|\right)$$

Thm 8,7 $f \neq 0$

$$\max_{\bar{\Omega}\times\bar{I}} |u| \leq \max\left(\max_{\Gamma\times\bar{I}} |g|, \max_{\bar{\Omega}} |v|\right) + \frac{r^2}{2d} \max_{\bar{\Omega}\times\bar{I}} |f|$$

where r is the radius of a ball containing Ω.

Proof in the book.

Thm 8,8

$$\begin{cases} \dot{u} - \Delta u = 0 & \mathbb{R}^d \times I \\ u(\cdot,0) = v & \mathbb{R}^d \end{cases}$$

has at most one bounded solution in $\mathbb{R}^d \times [0,T]$ for any T.

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without the assumption $|u| \le M$ there is a non-zero solution to the homogeneous problem ($v=0$)

$$u(x,t) = \sum_{n=0}^{\infty} f^{(n)}(t) \frac{x^{2n}}{(2n)!} \; ,$$

$$f(t) = e^{-1/t^2}, \quad t>0, \quad f(0)=0,$$

$$u''_{xx} = \sum_{n=0}^{\infty} f^{(n)}(t)\, 2n(2n-1) \frac{x^{2n-2}}{(2n)!} =$$

$$= \sum_{n=1}^{\infty} f^{(n)}(t) \frac{x^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} f^{(n+1)}(t) \frac{x^{2n}}{(2n)!}$$

$$= u'_t \quad \therefore \quad \dot{u} - u_{xx} = 0 .$$

$u(x,0)=0$ since $f^{(n)}(0)=0$ because e^{-1/t^2} goes to zero quickly enough.

FEM for parabolic problems

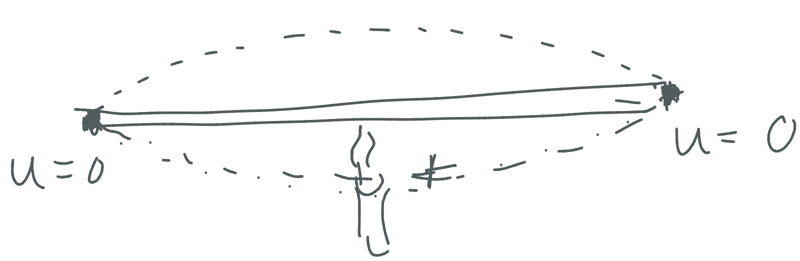
The semi-discrete Galerkin FEM

Let Ω be convex in $\mathbb{R}^2$ with smooth boundary.

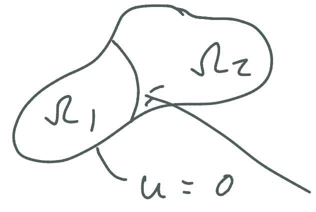
$$\begin{cases} \dot{u} - \Delta u = f, & \Omega \times I \\ u = 0, & \Gamma \times I \\ u(\cdot, 0) = v, & \text{in } \Omega \end{cases}$$

On weak form: find $u(t) \in H^1_0(\Omega)$ s.t.

$$\begin{cases} (\dot{u}, \phi) + a(u, \phi) = (f, \phi) \quad \forall \phi \in H^1_0(\Omega) \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ u(0) = v \end{cases}$$



Problem 3.3

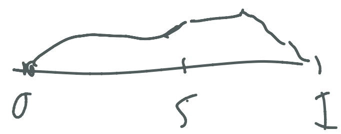


$-a_1 \Delta u_1 = f_1, \ \Omega_1$

$-a_2 \Delta u = f_2, \ \Omega_2$

$S: \quad u_1 = u_2$

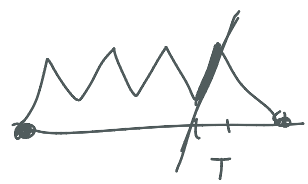
$a_1 n \cdot \nabla u_1 = a_2 n \cdot \nabla u_2$



Let T_h triangulation of Ω

$$V_h = \{ v \in C(\Omega) : v|_T \text{ affine } \forall T \in T_h, v|_\Gamma = 0 \}$$

$V_h \subset H_0^1(\Omega)$



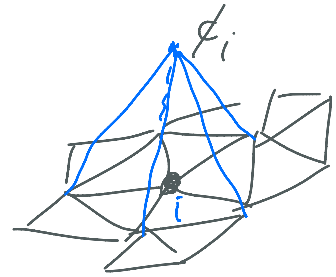
Semi-discrete FEM

$$\begin{cases} (\dot{u}_h, \phi) + a(u_h, \phi) = (f, \phi) \quad \forall \phi \in V_h \\ u_h(0) = v_h \end{cases}$$

$a(v, w) = (\nabla v, \nabla w)$, v_h approximates v.

Let $V_h = \text{span}(\{\phi_i\}_{i \in N})$

N: the set of interior nodes



We can write $u_h = \sum_{j \in N} \alpha_j(t) \phi_j$

As test function we pick $\phi = \phi_i$, $i \in N$

$$\sum_{j \in N} \alpha_j'(t)(\phi_j, \phi_i) + \alpha_j(t)\, a(\phi_j, \phi_i) = (f, \phi_i)$$

$i \in N$.

$\alpha_i(0) = \gamma_i \quad i \in N$ where

$$v_h = \sum_{i \in N} \gamma_i \phi_i$$

Let $M_{ij} = (\phi_j, \phi_i)$

$K_{ij} = a(\phi_j, \phi_i) \qquad b_i(t) = (f, \phi_i)$

$$* \begin{cases} M\dot{\alpha} + K\alpha = b \\ \alpha(0) = \gamma \end{cases}$$ system of ODE.

M and K are sym $\underbrace{\text{positive definite}}_{\text{SPD}}$
and invertable

There is a unique solution $\alpha(t)$ to
$*$.

<u>Stability result for u_h.</u>

$\underline{\frac{1}{2}\frac{\partial}{\partial t}(u_h^2) = u_h \dot{u}_h}$

Let $\phi = u_h$

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + |u_h|^2_{H^1(\Omega)} = \int_\Omega \dot{u}_h \cdot u_h\, dx + \int_\Omega \nabla u_h \cdot \nabla u_h\, dx$$

$|v|_{H^1} = \|\nabla v\|_{L^2}$

$$= (f, u_h) \overset{C.S}{\leq} \|f\| \cdot \|u_h\| \overset{P.F}{\leq} \underbrace{C\|f\|}_{a} \cdot \underbrace{|u_h|_{H^1}}_{b}$$

$$\leq C\|f\|^2 + \frac{1}{2}|u_h|^2_{H^1}$$

$$\frac{d}{dt}\|u_h\|^2 + |u_h|^2_{H^1} \leq C\|f\|^2$$

$$\frac{1}{2}\frac{d}{dt}\underbrace{\|u_h\|^2} = \|u_h\| \cdot \frac{d}{dt}\|u_h\|$$

$$\cancel{\|u_h\|} \cdot \frac{d}{dt}\|u_h\| \leq \|f\| \cdot \cancel{\|u_h\|}$$

$$(**)\ \|u_h(t)\| \leq \|v_h\| + \int_0^t \|f(s)\|\, ds$$

Error bound

Thm 10.1

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq \|v_h - v\|_{L^2(\Omega)} +$$

$$+ Ch^2\left(\|v\|_{H^2(\Omega)} + \int_0^t \|\dot{u}\|_{H^2(\Omega)} ds\right)$$

Proof: Ritz projection: $R_h : H^1_0(\Omega) \to V_h$

$$a(R_h z, w) = a(z, w) \quad w \in V_h$$

By elliptic regularity $\|z - R_h z\|_{L^2(\Omega)} \leq Ch^2\|z\|_{H^2}$

$$z \in H^2 \cap H^1_0$$

$$u_h - u = \underbrace{u_h - R_h u}_{\theta(t) \in V_h} + \underbrace{R_h u - u}_{\rho(t)}$$

$\rho(t)$) $\|\rho(t)\|_{L^2} \le Ch^2 \|u\|_{H^2(\Omega)} \le Ch^2 \|v + \int_0^t \dot{u}\, ds\|_{H^2}$

$$\le Ch^2 \|v\|_{H^2} + Ch^2 \int_0^t \|\dot{u}\|_{H^2}\, ds$$

$\theta(t)$) $\theta(t) \in V_h$, $w \in V_h$

$$(\dot{\theta}, w) + a(\theta, w) = \underline{(\dot{u}_h, w)} - (R_h \dot{u}, w)$$

$$+ \underline{a(u_h, w)} - a(R_h u, w)$$

$$= (f, w) - (R_h \dot{u}, w) - a(u, w)$$

$$= (\dot{u} - R_h \dot{u}, w) = -(\dot{\rho}(t), w)$$

Since $(\dot{\theta}, w) + a(\theta, w) = (-\dot{\rho}(t), w)$

$\theta \in V_h$ $\qquad \forall w \in V_h$

Then $\|\theta(t)\| \le \underline{\|\theta(0)\|} + \int_0^t \underline{\|\dot{\rho}(s)\|}\, ds$

$$\|\theta(0)\| = \|v_h - R_h v\| \le \|v_h - v\| + \|\underline{v - R_h v}\|$$

$$\le \|v_h - v\| + Ch^2 \|v\|_{H^2}$$

$$\|\dot{\rho}\| \le \|R_h \dot{u} - \dot{u}\| \le Ch^2 \|\dot{u}\|_{H^2}$$

$$\|u_h - u\| \le \|v_h - v\| + Ch^2 \|v\|_{H^2} + \int_0^t \|\dot{u}(s)\|_{H^2}\, ds \cdot Ch^2 \quad \blacksquare$$