Thm 10.2

$$|u(t)-u_h(t)|_{H^1(\Omega)} \le |v_h - v|_{H^1(\Omega)}$$
$$+ Ch\left(\|v\|_{H^2(\Omega)} + \|u(t)\|_{H^2(\Omega)} + \left(\int_0^t \|\dot u\|^2_{H^1(\Omega)} ds \right)^{1/2} \right)$$

Proof: $u_h - u = \underbrace{u_h(t) - R_h u(t)}_{\theta(t) \in V_h} + \underbrace{R_h u(t) - u(t)}_{\rho(t)}$

We have $|\rho(t)|_{H^1(\Omega)} \le Ch\|u\|_{H^2(\Omega)}$

We recall $\cancel{\theta(t)\in} V_h$

$$(\dot\theta, w) + a(\theta, w) = (-\dot\rho, w) \quad \forall w \in V_h$$

We let $w = \dot\theta$

$$\|\dot\theta\|^2 + \frac{1}{2}\frac{\partial}{\partial t}|\theta|^2_{H^1} = (\dot\theta, \dot\theta) + a(\theta, \dot\theta)$$

$$\left(\int_\Omega \nabla\theta \cdot \nabla\dot\theta = \int_\Omega \frac{1}{2}\frac{\partial}{\partial t} \nabla\theta \cdot \nabla\theta \right)$$

$$= (-\dot\rho, \dot\theta) \overset{C.S}{\le} \|\dot\rho\| \cdot \|\dot\theta\| \le \frac{1}{2}\|\dot\rho\|^2 + \frac{1}{2}\|\dot\theta\|^2$$

$$\left(ab \le \varepsilon^{1/2} a \cdot \varepsilon^{-1/2} b \le \frac{1}{2}\varepsilon a^2 + \frac{1}{2\varepsilon} b^2 \right)$$

$$\Rightarrow \cancel{\|\dot\theta\|^2} + \frac{\partial}{\partial t}|\theta|^2_{H^1(\Omega)} \le \|\dot\rho\|^2$$

$$|\theta(t)|^2_{H^1(\Omega)} - |\theta(0)|^2_{H^1(\Omega)} \le \int_0^t \|\dot\rho(s)\|^2 ds$$

$$\text{or} \quad |\theta(t)|^2_{H^1(\Omega)} \le |v_h - R_h v|^2_{H^1(\Omega)} + \int_0^t \|\dot\rho(s)\|^2 ds$$

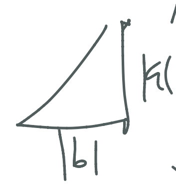
$$\le |v_h - v|^2_{H^1(\Omega)} + Ch^2\|v\|_{H^2(\Omega)} +$$

$$+ \int_0^t \|\dot u - R_h \dot u\|^2 ds \le$$

$$\le \quad - \,\|\, - \quad + Ch^2 \int_0^t \|\dot u\|^2_{H^1(\Omega)} ds$$

$$|u_h - u(t)|^2_{H^1(\Omega)} \le |v - v_h|^2_{H^1(\Omega)} + Ch^2\|v\|^2_{H^2(\Omega)} + Ch^2\|u(t)\|^2_{H^2(\Omega)}$$

$$+ Ch^2 \int_0^t \|\dot u\|^2_{H^1(\Omega)} ds \qquad \sqrt{a^2+b^2} \le |a| + |b|$$



$$|u_h - u(t)|_{H^1(\Omega)} \le |v - v_h|_{H^1(\Omega)} + Ch\|v\|_{H^2(\Omega)}$$

$$+ Ch\|u(t)\|_{H^2(\Omega)} + Ch\left(\int_0^t \|\dot u\|^2_{H^1(\Omega)} ds\right)^{1/2}$$

Note that $|v_h - v|_{H^1(\Omega)}$ can be bounded

$v_h = I_h v$ $\quad |v_h - v|_{H^1(\Omega)} \le Ch\|v\|_{H^2}$.

$= R_h v$

Thm 10.3 If $f=0$ $v_h = P_h v$,

where $P_h : L^2(\Omega) \to V_h$ is defined by

$$\int_\Omega P_h v \cdot w \, dx = \int_\Omega v \cdot w \, dx \quad \forall w \in V_h$$

$v \in L^2(\Omega)$

Then $\|u_h(t) - u(t)\|_{L^2(\Omega)} \le C h^2 t^{-1} \|v\|_{L^2(\Omega)}$

Fully discrete FD in time FEM in space

Backward Euler Galerkin method

Time interval $[0, T]$, $k = \frac{T}{N}$,

N is number of time steps, k is timestep

$$\bar{\partial}_t U^n = \frac{U^n - U^{n-1}}{k} \quad \text{discrete time derivative}$$

BE Galerkin: find $\{U^n\}_{n=1}^N \in V_h$ s.t.

$$\begin{cases} (\bar{\partial}_t U^n, w) + a(U^n, w) = (f(t_n), w) & \forall w \in V_h, \ n \geq 1 \\ U^0 = v_h & \end{cases}$$

Let $V_h = \text{span}(\{\phi_i\})$

We let $K_{ij} = a(\phi_j, \phi_i)$ $M_{ij} = (\phi_j, \phi_i)$

$b_j^n = (f(t_n), \phi_j)$, $U^n = \sum_{i \in N} \alpha_i^n \phi_i$

We get $\dfrac{M\alpha^n - M\alpha^{n-1}}{k} + K\alpha^n = b^n$ or

$$\underbrace{(M + k\cdot K)}_{\text{SPD} \Rightarrow \text{invertable}} \alpha^n = M\alpha^{n-1} + k b^n, \quad n \geq 1$$

Thm $\|U^n\| \leq \|U^0\| + k \sum_{j=1}^{n} \|f^j\|$, $f^j = f(t_j)$

Proof Let $w = U^n \in V_h$

$(\bar{\partial}_t U^n, U^n) + a(U^n, U^n) = (f^n, U^n)$, multiply w. k

$\Rightarrow \|U^n\|^2 - (U^{n-1}, U^n) + k|U^n|^2_{H^1(\Omega)} \overset{C.S}{\leq} k\|f^n\| \|U^n\|$

so $\|U^n\|^2 \leq (\|U^{n-1}\| + k\|f^n\|)\|U^n\|$.

$$\|U^n\| \leq \|U^{n-1}\| + k\|f^n\| \leq$$

$$\leq \|U^{n-2}\| + k\|f^{n-1}\| + k\|f^n\| +$$

$$\leq \|U^0\| + k \sum_{j=1}^{n} \|f^j\| \quad \blacksquare$$

Thm 10.5: If v_h is chosen so that

$$\|v_h - v\| \le Ch^2 \|v\|_{H^2} \quad \text{then}$$

$$\|U^n - u(t_n)\| \le Ch^2\left(\|v\|_{H^2(\Omega)} + \int_0^{t_n} \|\dot u\|_{H^2(\Omega)} ds\right) + Ck\int_0^{t_n} \|\ddot u\| ds$$

Proof: $$U^n - u(t_n) = \underbrace{U^n - R_h u(t_n)}_{\theta^n \in V_h} + \underbrace{R_h u(t_n) - u(t_n)}_{\rho^n}$$

As before

$$\|\rho^n\| \le Ch^2 \|\overset{u(0)+\int_0^{t_n} \dot u \, ds}{u(t_n)}\|_{H^2(\Omega)} \le Ch^2\left(\|v\|_{H^2(\Omega)} + \int_0^{t_n} \|\dot u\|_{H^2} ds\right)$$

$$(\bar\partial_t \theta^n, w) + a(\theta^n, w) = (\bar\partial_t U^n, w) + a(U^n, w) - (\bar\partial_t R_h u(t_n), w) - a(R_h u(t_n), w)$$

$$= (f(t_n), w) - (\bar\partial_t R_h u(t_n), w) - a(u(t_n), w)$$

$$= (\dot u(t_n) - \bar\partial_t R_h u(t_n), w) \quad \forall w \in V_h$$

We have

$$R_h \bar\partial_t u(t_n) - \dot u(t_n) = (R_h - I)\bar\partial_t u(t_n) + (\bar\partial_t u(t_n) - \dot u(t_n)) = \omega_1^n + \omega_2^n$$

Stability of θ^n gives

$$\|\theta^n\| \le \|\theta^0\| + k\sum_{j=1}^{n} \|\omega_1^j\| + k\sum_{j=1}^{n} \|\omega_2^j\|$$

$$\|\theta^0\| = \|v_h - R_h v\| \le \|v_h - v\| + \|v - R_h v\| \le$$

$$\le C h^2 \|v\|_{H^2(\Omega)}$$

$$k\sum_{j=1}^{n} \|\omega_1^j\| \le k \sum_{j=1}^{n} \|(R_h - I)\underbrace{\bar\partial_t u(t_j)}_{\frac{u(t_j)-u(t_{j-1})}{k} = \frac{1}{k}\int_{t_{j-1}}^{t_j} \dot u(s)\,ds}\| \le$$

$$\le \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} C h^2 \|\dot u\|_{H^2(\Omega)}\, ds =$$

$$= \int_0^{t_n} C h^2 \|\dot u\|_{H^2(\Omega)}\, ds .$$

$$k\sum_{j=1}^{n} \|\omega_2^j\| = k\sum_{j=1}^{n} \|\underbrace{\bar\partial_t u(t_j) - \dot u(t_j)}\|$$

$$\left\{ \begin{aligned} \omega_2^j &= \frac{u(t_j)-u(t_{j-1})}{k} - \dot u(t_j) \\ &= -\frac{1}{k} \int_{t_{j-1}}^{t_j} (s-t_{j-1})\,\ddot u(s)\,ds \end{aligned} \right\}$$

$$\le \sum_{j=1}^{n} \left\| \int_{t_{j-1}}^{t_j} (\underbrace{s-t_{j-1}}_{\le k})\,\ddot u(s)\,ds \right\| \le k\int_0^{t_n} \|\ddot u(s)\|\,ds$$

$\square$

The Crank-Nicolson Galerkin method

find $\{U^n\}_{n=1}^{N} \in V_h$ s.t.

$$\begin{cases} \left(\bar{\partial}_t U^n, w\right) + a\left(\frac{U^n + U^{n-1}}{2}, w\right) = \left(f(t_{n-\frac{1}{2}}), w\right) & w \in V_h \\ & n \geq 1 \\ U^0 = v_h & \end{cases}$$

Given U^{n-1}

$$\underbrace{\left(M + \frac{k}{2}K\right)}_{SPD} \alpha^n = \left(M - \frac{k}{2}K\right)\alpha^{n-1} + k b^{n-\frac{1}{2}}$$

$$U^n = \sum_{j \in N} \alpha_j^n \phi_j$$