Stability of Crank-Nicolson

Thm $\|U^n\| \le \|U^0\| + k\sum_{j=1}^{n} \|f^{j-\frac{1}{2}}\|$

Proof: Let $w = U^n + U^{n-1}$ in

$$\left(\bar{\partial}_t U^n, w\right) + a\left(\frac{U^n + U^{n-1}}{2}, w\right) = \left(f^{n-\frac{1}{2}}, w\right)$$

$$\frac{1}{k}\left(\|U^n\|^2 - \|U^{n-1}\|^2\right) = \frac{1}{k}\left(U^n - U^{n-1}, U^n + U^{n-1}\right)$$

$$\le \left(\bar{\partial}_t U^n, U^n + U^{n-1}\right) + a\left(\frac{U^n + U^{n-1}}{2}, U^n + U^{n-1}\right)$$

$$= \left(f^{n-\frac{1}{2}}, U^n + U^{n-1}\right) \overset{C.S.}{\le} \|f^{n-\frac{1}{2}}\| \cdot \|U^n + U^{n-1}\|$$

$$\le \|f^{n-\frac{1}{2}}\| \cdot \left(\|U^n\| + \|U^{n-1}\|\right)$$

$$\Rightarrow \frac{1}{k}\left(\|U^n\| - \|U^{n-1}\|\right)\cancel{\left(\|U^n\| + \|U^{n-1}\|\right)} \le$$

$$\le \|f^{n-\frac{1}{2}}\| \cdot \cancel{\left(\|U^n\| + \|U^{n-1}\|\right)}$$

$$\|U^n\| \le \|U^{n-1}\| + k\|f^{n-\frac{1}{2}}\| \le$$

$$\le \|U^0\| + k\sum_{j=1}^{n} \|f^{j-\frac{1}{2}}\|$$

∎

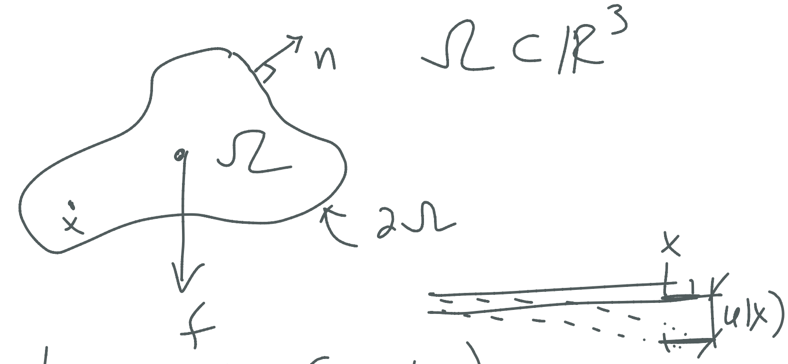
Thm 10.6 If v_h is chosen so that

$$\|v - v_h\| \le Ch^2 \|v\|_{H^2} \text{ then}$$

$$\|U^n - u(t_n)\| \le Ch^2\left(\|v\|_{H^2(\Omega)} + \int_0^{t_n} \|\dot{u}\|_{H^2(\Omega)} ds\right) + Ck^2 \int_0^{t_n} \left(\|u_t^{(3)}\| + \|\Delta u_t^{(2)}\|\right) ds$$

Solid mechanics

Small deformations of an elastic body $\Rightarrow$ equations of linear elasticity



- $u(x)$ displacement (vector)
- $\sigma(u)$ stress (matrix 3×3) force/area

  $\int_{\partial\omega} \sigma \cdot n \, ds$ is surface force on subvolume $\omega \subset \Omega$.
- f body force (i.e. gravity) (vector)

Equilibrium of forces on any $w \subset \Omega$

$$0 = \int_w f \, dx + \int_{\partial w} \sigma \cdot n \, ds \overset{\text{Gauss}}{=} \int_w f + \nabla \cdot \sigma \, dx$$

$$\therefore \quad -\nabla \cdot \sigma = f \quad \forall x \in \Omega$$

Cauchy's equilibrium equation.

σ is symmetric. It has 6 unknowns

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

The strain $\varepsilon(u)$ is a measure of deformation

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$ i.e. symmetric 3x3 matrix

Ideally ε should be invariant under rigid body motions (RBM) which are translation and rotation

As defined here it is invariant under translation and linearized (small) rotation.

X.3

For isotropic materials Hooke's law

$$\sigma(u) = 2\mu\,\varepsilon(u) + \lambda(\nabla\cdot u) I \quad , \quad I \text{ is } 3\times 3 \text{ identity}$$

and μ, λ are the Lamé parameters (material parameters)

We get the following system

$$\text{Dirichlet} \rightarrow \begin{cases} -\nabla\cdot\sigma = f \\ \sigma = 2\mu\,\varepsilon(u) + \lambda(\nabla\cdot u) I \\ u = 0 \quad \text{on } \Gamma_D \\ \sigma\cdot n = g_N \quad \text{on } \Gamma_N \end{cases}$$

unknowns are

$\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}$

u_1, u_2, u_3

Weak formulation

Let $V = \{ v \in (H^1(\Omega))^3 : v|_{\Gamma_D} = 0 \}$.

V is a Hilbert space.

We multiply by a test function and $\int_\Omega$

$$(f, v) = \int_\Omega f_1 v_1 + f_2 v_2 + f_3 v_3 \, dx = (-\nabla\cdot\sigma, v)$$

$$= -\int_\Omega \left[\frac{\partial}{\partial x_1} \; \frac{\partial}{\partial x_2} \; \frac{\partial}{\partial x_3} \right] \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} dx$$

$$= -\int_\Omega \left[\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \quad \cdots \right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} dx$$

$$= -\int_\Omega \sum_{i,j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, dx = -\sum_{i,j=1}^{3} \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, dx =$$

$$= \sum_{i,j=1}^{3} -(\sigma_{ij}, n_j v_i)_{\partial\Omega} + \left(\sigma_{ij}, \frac{\partial v_i}{\partial x_j}\right).$$

Let $A : B = \sum_{i,j=1}^{3} A_{ij} B_{ij}$ and $v|_{\Gamma_D} = 0$

$$(\sigma : \nabla v) = (f, v) + (g_N, v)_{\Gamma_N}$$

If A is symmetric and B anti-symmetric with zero diagonal $B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix}$

then $A : B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} : \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix} = 0$

Check

$$\nabla v = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right) + \frac{1}{2}\left(\nabla v - (\nabla v)^T\right)$$

So $\sigma : \nabla v = \sigma(u) : \varepsilon(v)$

i.e. $(\sigma(u) : \varepsilon(v)) = (f, v) + (g_N, v)_{\Gamma_N}$

Hooke's law

Find $u \in V$ such that

$$2\mu(\varepsilon(u), \varepsilon(v)) + \lambda(\nabla\cdot u, \nabla\cdot v) = (f, v) + (g_N, v)_{\Gamma_N}$$

for all $v \in V$.

With abstract notation: find $u \in V$ s.t.

$a(u, v) = L(v) \quad \forall v \in V$ where

$$a(u,v) = 2\mu(\varepsilon(u), \varepsilon(v)) + \lambda(\nabla\cdot u, \nabla\cdot v)$$

$$L(v) = (f, v) + (g_N, v)_{\Gamma_N} \qquad L: V \to \mathbb{R}$$

<u>Existence and uniqueness</u>

Let $\|b\|_V^2 = \sum_{i=1}^{3} \|b_i\|^2_{H^1(\Omega)}$, $\left(\|A\|_V^2 = \sum_{i,j=1}^{3} \|A_{ij}\|^2_{H^1}\right)$

To apply Lax-Milgram we need to show boundedness of a and L and coercivity of a.

$$\begin{aligned}
a(u,v) &= 2\mu(\varepsilon(u), \varepsilon(v)) + \lambda(\nabla\cdot u, \nabla\cdot v) \\
&\overset{C.S}{\leq} 2\mu\|\varepsilon(u)\|\cdot\|\varepsilon(v)\| + \lambda\|\nabla\cdot u\|\cdot\|\nabla\cdot v\| \\
&\leq C\|u\|_V\cdot\|v\|_V + \lambda\|u\|_V\|v\|_V \\
&\leq C\|u\|_V\cdot\|v\|_V
\end{aligned}$$

$$|L(v)| = |(f,v) + (g_N, v)_{\Gamma_N}| \leq$$

$$\leq \|f\| \cdot \|v\| + \|g_N\|_{L^2(\Gamma_N)} \cdot \|v\|_{L^2(\Gamma_N)}$$ $\partial\Omega$

trace

$$\leq \|f\| \cdot \|v\|_V + \|g_N\|_{L^2(\Gamma_N)} \cdot \|v\|_V$$

$$\leq C\left(\|f\| + \|g_N\|_{L^2(\Gamma_N)}\right)\|v\|_V .$$

<u>Thm</u> (Korn's inequality)

$$C\|\nabla v\| \leq \|\varepsilon(v)\| \quad \forall v \in V$$

<u>Proof</u>: Assume $\Gamma_D = \partial\Omega$ i.e $u=0$ on $\partial\Omega$.

$$\|\varepsilon(v)\|^2 = \int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij}(v)\,\varepsilon_{ij}(v)\,dx =$$

$$= \int_\Omega \sum_{i,j=1}^{3} \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \cdot \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx$$

$$= \frac{1}{4}\int_\Omega \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j}\right)^2 + 2\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \left(\frac{\partial v_j}{\partial x_i}\right)^2 dx$$

$$= \underline{\frac{1}{2}\|\nabla v\|^2} + \frac{1}{2}\sum_{i,j=1}^{3}\int_\Omega \left(\frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i}\right) dx$$

need to show ≥ 0

$$\sum_{i,j=1}^{3} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} dx = -\sum_{i,j=1}^{3} \int_\Omega v_i \frac{\partial^2 v_j}{\partial x_i \partial x_j} + \cancel{\int_{\partial\Omega} n_j v_i \frac{\partial v_j}{\partial x_i} ds}$$

$$= \sum_{i,j=1}^{3} \int_\Omega \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} dx - \cancel{\int_{\partial\Omega} n_i v_i \frac{\partial v_j}{\partial x_j} ds}$$

$$= \int_\Omega \left(\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \right) \cdot \left(\sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} \right) dx =$$

$$= \int_\Omega (\nabla \cdot v)^2 dx \geq 0$$

The coercivity follows by Korn

$$a(v,v) = 2\mu \|\varepsilon(v)\|^2 + \lambda \|\nabla \cdot v\|^2 \geq 2\mu \|\varepsilon(v)\|^2$$
$$\geq C \|\nabla v\|^2 \geq m \|v\|_V^2 \quad v \in V.$$

Therefor Lax-Milgram guarantees existence of unique solution $u \in V$.

Furthermore

$$\|u\|_V^2 \leq C a(u,u) = C L(u) \leq$$
$$\leq C \left(\|f\| + \|g_N\|_{L^2(\Gamma_N)} \right) \|u\|_V$$

$$\therefore \; \|u\|_V \le C\left(\|f\| + \|g_N\|_{L^2(\Gamma_N)}\right)$$