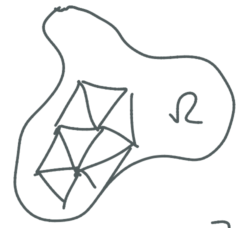
Finite element method

Let $\mathcal{T}_h$ be triangulation of Ω



$\bigcup_{T\in\mathcal{T}_h} T = \Omega$, $\{T\}$ are disjoint.

$$V_h = \{ v \in V : v|_T \in P_1(T)^3 , \forall T \in \mathcal{T}_h , v|_{\Gamma_D} = 0 \}$$

$$V = \{ v \in (H^1(\Omega))^3 : v|_{\Gamma_D} = 0 \}$$

V_h is the space of FE displacement vectors.

We get $I_h : (H^1(\Omega))^3 \to V_h$

$$\| v - I_h v \|_{\underset{(H^1)^3}{V}} \le C h |v|_{H^2(\Omega)}$$

<u>FEM</u>: find $u_h \in V_h$ s.t.

$$a(u_h, v) = L(v) \quad \forall v \in V_h$$

$$a(v,w) = 2\mu(\varepsilon(v), \varepsilon(v)) + \lambda(\nabla\cdot v, \nabla\cdot w)$$

$$L(w) = (f, w) + (g_N, w)_{\Gamma_N}$$

<u>Thm</u> If $u \in (H^2(\Omega))^3$ then

$$\| u - u_h \|_V \le C h |u|_{(H^2(\Omega))^3}$$

Proof: Let $e = u - u_h$

$$m\|e\|_V^2 \underset{\substack{\uparrow \\ \text{coercivity}}}{\le} a(e,e) = a(e, u - \underbrace{u_h}_{V_h}) \overset{\text{G.O.}}{=} a(e, u - I_h u)$$

$$\le C\|e\|_V \cdot \|u - \tilde{I}_h u\|_V \le C' h |u|_{H^2} \cdot \|e\|_V$$

$$\therefore \ \|e\|_V \le C h |u|_{(H^2(\Omega))^3} \quad \blacksquare$$

Implementation:

$$\sigma = [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{12} \ \sigma_{23} \ \sigma_{31}]^T$$

$$\varepsilon = [\varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{33} \ 2\varepsilon_{12} \ 2\varepsilon_{23} \ 2\varepsilon_{31}]^T$$

Hooke's law can be written as

$$\sigma = D\varepsilon$$

$$D = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Therefore $\varepsilon : \sigma = \varepsilon^T D \varepsilon$

$$a(u_h, v) = \int_\Omega \varepsilon(v)^T D\, \varepsilon(u_h)\, dx$$

$$\begin{bmatrix} u_{h,1}(x_i) \\ u_{h,2}(x_i) \\ u_{h,3}(x_i) \end{bmatrix} = \begin{bmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \end{bmatrix}$$

$$\{x_i\}_{i=1}^{n}$$

$$u_h = \begin{bmatrix} u_{h,1} \\ u_{h,2} \\ u_{h,3} \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 & 0 & \varphi_2 & 0 & \cdots \\ 0 & \varphi_1 & 0 & 0 & \varphi_2 & \cdots \\ 0 & 0 & \varphi_1 & 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ d_{21} \\ d_{22} \\ d_{23} \\ \vdots \\ d_{n1} \\ d_{n2} \\ d_{n3} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ \partial & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_{h,1} \\ u_{h,2} \\ u_{h,3} \end{bmatrix}$$

$$\boxed{\varepsilon = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)}$$

$$\varphi$$

$$B = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & \partial \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} \varphi_1 & & & \varphi_2 & & \cdots \\ & \varphi_1 & & & \varphi_2 & \cdots \\ & & \varphi_1 & & & \end{bmatrix}$$

$$\begin{bmatrix} d_{11} \\ d_{12} \\ d_{13} \\ \vdots \end{bmatrix}$$

$$\varepsilon = B\, d$$

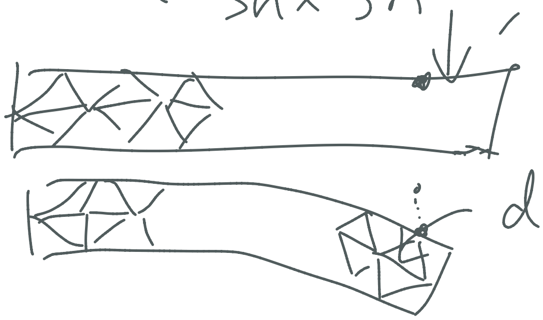
$$\sigma = D\, B\, d$$

FEM: $\left(\int_\Omega B^T D B \, dx \right) d = \int_\Omega \varphi^T f \, dx + \int_{\Gamma_N} \varphi^T g_N \, ds$

or $\quad K d = F \leftarrow 3n \times 1$

$$K = \int_\Omega B^T D B \, dx, \quad F = \int_\Omega \varphi^T f \, dx + \int_{\Gamma_N} \varphi^T g_N \, ds$$

$\uparrow$ $3n \times 3n$ $\quad$ n is number of free nodes.



Thermoelasticity

Mechanical and thermal strain $\Rightarrow$ Hooke's law can be generalized

$$\sigma(u,\theta) = 2\mu\varepsilon(u) + \lambda(\nabla\cdot u) - \alpha(\theta - \theta_0) I$$

$\uparrow$ coefficient

$$\begin{cases} -\nabla\cdot\left(2\mu\varepsilon(u) + \lambda\nabla\cdot u - \alpha\theta I\right) = f \\ u|_\Gamma = 0 \end{cases}$$

$\uparrow$ displacement

$$\begin{cases} \dot{\theta} - \nabla\cdot(\kappa\nabla\theta) + \alpha\nabla\cdot\dot{u} = g \\ \theta|_\Gamma = \theta_0 \end{cases}$$

$\uparrow$ temperature

let θ_1 solve $\dot{\theta}_1 - \nabla \cdot \kappa \nabla \theta_1 + 0 = g$

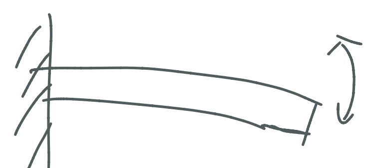
let θ_k solve $\begin{cases} \dot{\theta}_k - \nabla \cdot \kappa \nabla \theta_k + \alpha \nabla \cdot \dot{u}_{k-1} = g \\ -\nabla \cdot (2\mu \varepsilon(u_k) + \lambda \nabla \cdot u_k - \alpha \theta_k I) = f \end{cases}$

Existence of solution is given by linear theory + fix point argument Schauder fixpoint theorem.

Dynamics and model analysis

Newton's second law

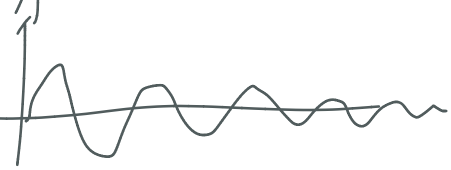
$$\rho \ddot{u} = f + \nabla \cdot \sigma(u)$$



Damping $+ D\dot{u}$

$$\rho \ddot{u} + \underline{D\dot{u}} = f + \nabla \cdot \sigma(u)$$

$$\boxed{M\ddot{d} = F - Kd}$$



If we assume $u = z\sin(\omega t)$ $f = 0$ $\rho = 1$

$$\boxed{-\nabla \cdot \sigma(z) = \omega^2 z} \leftarrow \text{eigenfunction}$$

On weak form: \ eigenfrequency \ eigenmodes

find $z \in V$ and $\omega^2 \in \mathbb{R}^+$ s.t

$(H^1)^3$

$$a(z,v) = \omega^2 (z,v) \quad \forall v \in V$$

$$\boxed{Kd = \omega^2 M d}$$ eigs