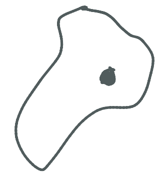
P 3.13

$$\begin{cases} -\nabla\cdot a_1\nabla u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\nabla\cdot a_2\nabla u_2 = f & \text{in } \Omega \\ u_2 = 0 & \text{on } \Gamma \end{cases}$$



$0 < a_0 \le a_1(x), a_2(x)$ smooth

Show $|u_1-u_2|_{H^1} \le \frac{C}{a_0^2}\|a_1-a_2\|_{L^\infty(\Omega)}\cdot\|f\|_{L^2(\Omega)}$

Find $u_1 \in H^1_0(\Omega)$: $(a_1\nabla u_1, \nabla v) = (f, v)\ \forall v\in V = H^1_0$

$u_2 \in H^1_0(\Omega)$: $(a_2\nabla u_2, \nabla v) = (f, v)\ \forall v\in V.$

$$a_0|u_1-u_2|^{\cancel{2}}_{H^1(\Omega)} \le (a_1\nabla(u_1-u_2), \nabla u_1-u_2) =$$

$$= (f, u_1-u_2) - (\underline{a_1}\nabla u_2, \nabla u_1-u_2) =$$

$$= \cancel{(f, u_1-u_2)} - \cancel{(a_2\nabla u_2, \nabla u_1-u_2)} - ((a_1-a_2)\nabla u_2, \nabla u_1-\nabla u_2)$$

$$\le \|a_1-a_2\|_{L^\infty(\Omega)}\|\nabla u_2\|_{L^2(\Omega)}\cdot|u_1-\cancel{u_2}|_{H^1(\Omega)}$$

Let $v = u_2$ $\quad \|a_2^{1/2}\nabla u_2\|^2 = (a_2\nabla u_2, \nabla u_2) = (f, u_2)$

$$\overset{C.S}{\le} \|f\|\cdot\|u_2\| \overset{P.F}{\le} \|f\|\cdot C|u_2|_{H^1}$$

$$a_0|u_2|^{\cancel{2}}_{H^1(\Omega)} \le \|a_2^{1/2}\nabla u_2\|^2 \le C\|f\|\cdot\cancel{|u_2|_{H^1}}$$

$$|u_2|_{H^1(\Omega)} \le \frac{C}{a_0}\|f\|$$

$$|u_1-u_2|_{H^1} \le \frac{C}{a_0^2}\|f\|\cdot\|a_1-a_2\|_{L^\infty(\Omega)}.$$

Ritz projection

$$\begin{cases} -\nabla\cdot a\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

$$a(u,v) = (a\nabla u, \nabla v) = (f,v) = L(v) \quad \forall v\in V = H_0^1(\Omega)$$

FEM: $a(u_h, v) = (f,v) = a(u,v) \quad \forall v \in V_h \subset V$

We define $R_h : H_0^1(\Omega) \to V_h$

$$a(R_h u, v) = a(u,v) \quad \forall v \in V_h \quad \text{i.e.}$$

$u_h = R_h u$ ..

$$a_0 |R_h u|^2_{H^1} \le a(R_h u, R_h u) = a(u, R_h u)$$

$$\le C |u|_{H^1} \cdot \cancel{|R_h u|_{H^1}}$$

$$\Rightarrow |R_h u|_{H^1(\Omega)} \le C |u|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

$$a_0 |u - R_h u|^2_{H^1(\Omega)} \le a(u - R_h u, u - R_h u) \overset{\text{G.O.}}{=} \quad -I_h u$$

$$\le C |\underline{u - R_h u}|_{H^1(\Omega)} \cdot |u - I_h u|_{H^1(\Omega)}$$

$$\le C h |u - R_h u|_{H^1(\Omega)} \cdot |u|_{H^2(\Omega)} \qquad \Omega \text{ convex}, \ a \text{ is } C^1$$

$$|u - R_h u|_{H^1(\Omega)} \le C h |u|_{H^2(\Omega)} \overset{\text{Elliptic Reg}}{\le} C h \|f\|_{L^2(\Omega)}$$

$-\nabla\cdot a\nabla u$

$\|u - R_h u\|_{L^2(\Omega)}$) let $e = u - R_h u$

adjoint problem $\begin{cases} -\nabla\cdot a\nabla\phi = e & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma \end{cases}$

$\phi \in V$: $(a\nabla w, \nabla \phi) = (w, e) \quad w \in V$ (H_0^1)

$$\|e\|^2 = (e,e) = (e, -\nabla\cdot a\nabla\phi) = (a\nabla e, \nabla\phi)$$

$$\le (a\nabla e, \nabla\phi - \hat{I}_h\phi) \le C|e|_{H^1} \cdot |\phi - I_h\phi|_{H^1}$$

$$\le \begin{cases} C(|u|_{H^1} + |R_h u|_{H^1})|\phi - I_h\phi|_{H^1} \le C|u|_{H^1} \cdot h|\phi|_{H^2} \\ Ch|u|_{H^2} \cdot |\phi - I_h\phi|_{H^1} \le C|u|_{H^2} h^2|\phi|_{H^2} \end{cases}$$

Assuming elliptic regularity Ω, a

$$|\phi|_{H^2(\Omega)} \le C\|e\| \qquad \left(\begin{aligned} \|D^2\phi\|_{L^2(\Omega)} &\le C\|-\nabla\cdot a\nabla\phi\|_{L^2(\Omega)} \\ &= C\|e\|_{L^2(\Omega)} \end{aligned} \right)$$

$$\|u - R_h u\| \le \begin{cases} Ch|u|_{H^1(\Omega)} \longleftarrow \text{ if } u \in H^2(\Omega) \\ \underline{Ch^2|u|_{H^2(\Omega)}} \end{cases}$$

Eigenvalue problems

$$\begin{cases} -\Delta\phi_i = \lambda_i\phi_i \\ \phi_i = 0 \end{cases}$$

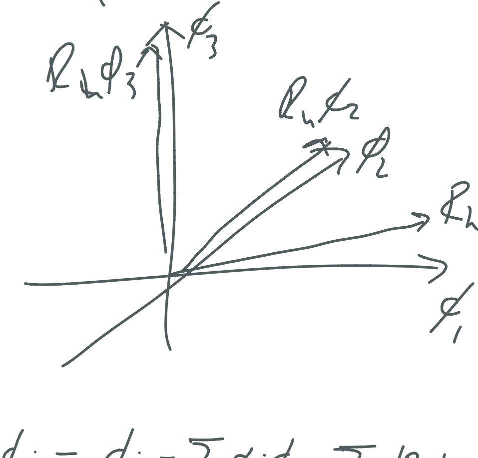
$$0 < \lambda_1 < \lambda_2 \le \dots$$

$$E_n = \text{span}(\{\phi_i\}_{i=1}^n)$$

$$|\lambda_n^h - \lambda_n|$$

$$R_h E_n = \{R_h v : v \in E_n\}$$

$$\dim(R_h E_n) = n \quad .$$



If $\dim(R_h E_n) < n$

Then $R_h\phi_j = \sum_{i\ne j} \alpha_i R_h\phi_i$

$$\phi_j - R_h\phi_j = \phi_j - \sum_{i\ne j}\alpha_i R_h\phi_i = \phi_j - \underline{\sum_{i\ne j}\alpha_i\phi_i} - \sum_{i\ne j}\alpha_i(R_h\phi_i - \phi_i)$$

or $\quad \underbrace{\phi_j - \sum_{(i \neq j)} \alpha_i \phi_i}_{L^2\text{-norm}} = \left(R_h \phi_j - \phi_j\right) - \sum_{(i \neq j)} \alpha_i \left(R_h \phi_i - \phi_i\right)$

$$\|\phi_j - \sum_{(i\neq j)} \alpha_i \phi_i\|^2 = \underbrace{\|\phi_j\|^2}_{\substack{\geq \\ =1}} + \|\sum_{(i \neq j)} \alpha_i \phi_i\|^2 \geq 1$$

So $\|R_h \phi_j - \phi_j - \sum_{(i\neq j)} \alpha_i (R_h \phi_i - \phi_i)\|_{L^2(\Omega)} \geq 1$

but $\|R_h \phi_j - \phi_j\| \leq C h^2 \lambda_j \qquad C_n h^2 \geq 1$

contradiction.

Thm 6.7 last step of proof

$$\lambda^n \leq \lambda_h^n \leq \lambda^n \frac{1}{1 - Ch^2}, \quad \alpha = Ch^2 < \frac{1}{2}$$

$\dfrac{1}{1-\alpha} \leq 1 + 2\alpha \quad$ since $(1+2\alpha)(1-\alpha) =$

$= 1 + \alpha - 2\alpha^2 = 1 + \alpha\underbrace{(1-2\alpha)}_{\geq 0} \geq 1$

$$\lambda_h^n \leq \lambda^n (1 + 2Ch^2)$$