### Example 2.32

In this example, we consider Hill's equation[6]

$$\ddot{y}(t) + a(t)y(t) = 0, \quad a(t+p) = a(t) \ \ \forall\, t \in \mathbb{R}, \tag{2.43}$$

[6] George William Hill (1838-1914), US American.

where $a$ is piecewise continuous and $p > 0$. Hill's equation describes an undamped oscillation with restoring force at time $t$ equal to $-a(t)y(t)$. The two-dimensional first-order system associated with (2.43) is given by

$$\dot{x}(t) = A(t)x(t), \quad A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \ \ \forall\, t \in \mathbb{R}. \tag{2.44}$$

Let $\Phi$ be the transition matrix function generated by $A$. Our intention is to apply Theorem 2.31 in the context of (2.44). To this end, we calculate the Floquet multipliers. Now,

$$\det(\lambda I - \Phi(p,0)) = \lambda^2 - \lambda \operatorname{tr} \Phi(p,0) + \det \Phi(p,0),$$

and, by statement (2) of Proposition 2.7,

$$\det \Phi(p,0) = \exp\left( \int_0^p \operatorname{tr} A(s) \mathrm{d}s \right) = 1.$$

Moreover, noting that $\Phi(t,0)$ is of the form

$$\Phi(t,0) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \dot{\varphi}_1(t) & \dot{\varphi}_2(t) \end{pmatrix} \ \ \forall\, t \in \mathbb{R},$$

where $\varphi_1$ and $\varphi_2$ are the unique solutions of (2.43) satisfying $\varphi_1(0) = 1 = \dot{\varphi}_2(0)$ and $\dot{\varphi}_1(0) = 0 = \varphi_2(0)$, respectively, it follows that

$$\operatorname{tr} \Phi(p,0) = \varphi_1(p) + \dot{\varphi}_2(p).$$

Consequently,

$$\det(\lambda I - \Phi(p,0)) = \lambda^2 - 2\gamma\lambda + 1, \quad \text{where} \quad \gamma := \tfrac{1}{2}(\varphi_1(p) + \dot{\varphi}_2(p)), \qquad (2.45)$$

and the Floquet multipliers are given by

$$\lambda_{\pm} = \gamma \pm \sqrt{\gamma^2 - 1}.$$

Invoking Theorem 2.31, we draw the following conclusions.

*Case 1:* $|\gamma| > 1$. Then $\lambda_+ > 1$ (if $\gamma > 1$) or $\lambda_- < -1$ (if $\gamma < -1$), and hence, at least one solution of (2.44) is unbounded on $\mathbb{R}_+$.

*Case 2:* $|\gamma| < 1$. Then $\lambda_{\pm} = \gamma \pm i\delta$ with $\delta > 0$. Since $\lambda_+\lambda_- = 1$, it follows that $|\lambda_+| = |\lambda_-| = 1$. Moreover, $\lambda_+$ and $\lambda_-$ are simple (and *a fortiori* semisimple) and hence all solutions of (2.44) are bounded on $\mathbb{R}_+$.

*Case 3:* $|\gamma| = 1$. Then $\gamma = \pm 1$ and $\lambda_+ = \lambda_- = \gamma$. All solutions of (2.44) are bounded on $\mathbb{R}_+$ if, and only if, $\gamma$ is semisimple. Since the algebraic multiplicity of $\gamma$ is two, $\gamma$ is semisimple if, and only if, $\ker(\gamma I - \Phi(p,0)) = \mathbb{C}^2$. Consequently,

$\gamma$ is semisimple if, and only if, $\Phi(p,0) = \gamma I$, that is, $\varphi_1(p) = \dot{\varphi}_2(p) = \gamma$ and $\dot{\varphi}_1(p) = \varphi_2(p) = 0$.

Irrespective of semisimplicity of $\gamma$, by Proposition 2.20, there exists at least one non-zero periodic solution of period $p$ if $\gamma = 1$ and of period $2p$ if $\gamma = -1$. Furthermore, we claim that, in the case of $\gamma$ being semisimple, every solution is $p$-periodic (if $\gamma = 1$) or $2p$-periodic (if $\gamma = -1$). To see this, assume that $\gamma$ is semisimple. Then the matrix

$$G := \begin{pmatrix} \log \gamma & 0 \\ 0 & \log \gamma \end{pmatrix}.$$

is a logarithm of $\Phi(p,0) = \gamma I$. By Theorem 2.30, there exists a piecewise continuously differentiable $p$-periodic function $\Theta : \mathbb{R} \to \mathbb{C}^{2\times 2}$ such that

$$\Phi(t,0) = \Theta(t) \exp(tp^{-1}G) \quad \forall\, t \in \mathbb{R}.$$

If $\gamma = 1$, then $G = 0$, and hence $\Phi(t,0) = \Theta(t)$ for all $t \in \mathbb{R}$, showing that $\Phi(t+p,0) = \Phi(t,0)$ for all $t \in \mathbb{R}$. Every solution $x$ of (2.44) is of the form $x(t) = \Phi(t,0)x(0)$ and is therefore $p$-periodic. If $\gamma = -1$, then

$$G = \begin{pmatrix} i\pi & 0 \\ 0 & i\pi \end{pmatrix},$$

whence

$$\Phi(t,0) = \Theta(t) \begin{pmatrix} e^{(i\pi/p)t} & 0 \\ 0 & e^{(i\pi/p)t} \end{pmatrix} \quad \forall\, t \in \mathbb{R}.$$

Therefore, $\Phi(t+2p,0) = \Phi(t,0)$ for all $t \in \mathbb{R}$, showing that every solution $x$ of (2.44) is $2p$-periodic.