Modelling project 1 for the course ODE and mathematical modelling MVE162/MMG551 in year 2021.
Two models of biological competition.

The modelling project assignment is obligatory and consists of a theoretical part that requires some mathematical reasoning and an implementation part including writing a simple code in Matlab or other programming language solving an ODE, graphical output, analysis of numerical solutions and conclusions.

Each of you must write an own individual report but you are encouraged to work in small groups of 2-3 people discussing theoretical and programming problems.

The report must we written in Swedish as a small scientific article that a person who did not study the course can understand. It must include: 1) theoretical argumentation, with necessary references 2) numerical results with graphical illustrations and 3) interpretation of theoretical and numerical results. Foreign students can write the report in English.

Upload your report and computer codes into both "math" and "writing" variant of the assignment for this Modelling project 1 in Canvas for checking separately mathematics and scientific writing. Names of the group members must be specified in your individual reports if a group activity took place.

Students will have a lecture on scientific writing by Elin Götmark on the 15-th of April and will get feedback on writing style aspects of their reports.

Grades for your reports for each modelling project will contribute 16% to the final marks for the course.

Logistic equation and two species competition model.

Let $x_i(t), i = 1, 2.$ be populations of two species. Each of the species grows with intrinsic growth rate r_i in case when infinite resources are available: $x_i' = r_i x_i$, $r_i > 0$.

Limited resources lead to competition within the population and a limited growth rate for the large size of the population: $r_i(1 - \frac{x_i}{K_i})$, $K_i > 0$. This model is called the logistic equation:

$$x_i' = r_i x_i \left(1 - \frac{x_i}{K_i}\right) \tag{1}$$

The competition between different species leads to a decrease in each population with the decreasing rate proportional to the competitor population size: $-\alpha_1 x_2$ for the population x_1 and $-\alpha_2 x_1$ for the population x_2 with competition coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$. The corresponding system of equations describes the evolution of two competing species:

$$\begin{aligned} x_1' &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \qquad (2) \\ x_2' &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_2 x_1 \end{aligned}$$

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Possible working plan and discussion questions

You should not think of the numbered list in the working plan below as sections of your paper. Instead, think about how best to organize your paper according to the lecture on scientific writing.

1. Describe all qualitatively different solutions to the equation (1) without solving the equation analytically and give a biological interpretation of your mathematical conclusions.

2. Illustrate your theoretical conclusions about solutions of (1) by a picture with several representative curves for different initial data.

3. Find formulas for all nullclines of the system (2) that are lines where $x_1' = 0$ or $x_2' = 0$. Find all stationary (equilibrium) points (they are intersections of nullclines).

4. Describe geometrically all possible qualitatively different relative positions of nullclines and equilibrium points (there are four distinct cases). Sketch pictures illustrating each case with nullclines marked by different colors. You can do it by hand or using Paint application in Windows. Give a biological interpretation of all possible equilibrium points.

5. Characterize all possible qualitatively different relative positions of nullclines analytically by a particular combination of parameters $\frac{r_1}{\alpha_1}$, K_1 and $\frac{r_2}{\alpha_2}$, K_2.

6. Investigate stability properties of all equilibrium points of (2) for each of the qualitatively different relative positions of nullclines. One can do it by applying the Grobman-Hartman theorem and the the criterion for classification of phase portraits around equilibrium points for linear systems in plane by the determinant and the trace of the systems matrix. One can also give an elementary purely geometric proof using properties of nullclines.

7. Draw pictures with phase portraits together with isoclines for each of the different cases above.

8. Give a biological interpretation to each of the four possible scenarios of evolution by (2).

9. What is qualitative difference between the behavior of solutions to the logistic equation and to the two species competition model when $t \to \infty$? How this difference depends on α_1 and α_2 if they are small or large (with other parameters fixed)?