**Bonus home assignment**
**- in ODE and Mathematical modeling MMG511/TMV162. Spring 2020.**
**Modeling pendulum with oscillating pivot.**

*This home assignment is voluntary and will give if successfully solved, 2 bonus points to the exam.*
*A short report must include:* ***1) theoretical argumentation 2) numerical results with graphical output and 3) interpretation of results.***

***Supply your reports (no revisions) together with Matlab codes with clear comments to Canvas latest on the 15-td of May.***

## Introduction.

Read literature about the Hill equation: pp. 55-57 in the course book. A copy of this text is available at Canvas.

A free pendulum without friction is described by the equation

$$l\theta'' = -g\sin\theta,$$

(see lecture notes) and has two stationary states: downward and upward: $\theta = 0$, $\theta' = 0$ and $\theta = \pi$, $\theta' = 0$ (plus infinitely many equivalent ones by $2\pi$ periodicity of the angle $\theta$)

The downward stationary state of the free pendulum is stable (pendulum stays close to it if we deviate it slightly). The upward stationary state of the free pendulum without friction is unstable (pendulum falls down if we deviate it slightly from the exact upward position).

The pendulum with vertically oscillating pivot is described by the equation with $\xi(t) = A\cos(\omega t)$ added for the coordinate of the oscillating pivot:

$$l\theta'' = -\left[g + \xi''(t)\right]\sin\theta,$$

and has a remarkable property. If the frequency $\omega$ of the pivot movement $\xi(t)$ is high enough the pendulum can "stabilize" around the upper state $\theta = \pi$, $\theta' = 0$.

Introducing a time scale $\tau$ where $\tau = \omega t$ we arrive to the equation in the more convenient form

$$\theta'' = -\left[\frac{g}{l\omega^2} - \frac{A}{l}\cos(\tau)\right]\sin\theta,$$

Both for theoretical analysis and for numerical solution one always rewrites second order equations as a system of two equations for $x_1(\tau) = \theta(\tau)$ and $x_2(\tau) = \theta'(\tau)$ :

$$\begin{aligned} x_1'(\tau) &= x_2(\tau) \\ x_2'(\tau) &= -\left[\frac{g}{l\omega^2} - \frac{A}{l}\cos(\tau)\right]\sin(x_1) \end{aligned}$$

It is convenient to introduce parameters $a = \frac{g}{l\omega^2}$ and $\varepsilon = \frac{A}{2l}$ leading to the system in the non-dimensional form:

$$\begin{aligned} x_1'(\tau) &= x_2(\tau) \\ x_2'(\tau) &= -\left[a - 2\varepsilon\cos(\tau)\right]\sin(x_1) \end{aligned}$$

Find the linearization of pendulum equations around the upward stationary state $x_1 = \pi$, $x_2 = 0$ to approximate movements of the pendulum close to the upper position. This linearization is a particular example of the Hill equation introduced on pp. 55-57 of the course book.

**Continuation is on the next page:**

For for the linearization around the upper equilibrium point $x_1 = \pi$, $x_1' = x_2 = 0$, and $l = 1$ answer following questions.

**1.** Apply the theory of the Hill equation on pp. 55-57 of the course book to to find a condition implying that a particular arbitrary combination $a = \frac{g}{l\omega^2}$ and $\varepsilon = \frac{A}{2l}$ corresponds to te situation when all solutions to the linearized system are bounded .

**2.** Write a Matlab code to calculate numerically the monodromy matrix $\Phi(p, 0)$ and it's trace for arbitrary fixed $a$ and $\varepsilon$.

**3.** Find numerically and draw using Matlab a domain in the plane $(a, \varepsilon)$ corresponding to bounded solutions.

What is approximately the smallest frequency $\omega$ implying boundedness of solutions for the amplitude $A = 0.01$ of the pivot oscillations?

**4.** Illustrate your conclusions by typical orbits of the linearized equation.

**5.** Make comments about the physical meaning of your results.

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One can try to investigate boundedness of solutions for the linearization around the downward stationary state (not required here!).

For particular frequencies $\omega$ even for small amplitudes $A$ the downward orientation of the pendulum can become unstable showing movements with the amplitude of movement much higher then $A$. It is similar to how children with a small but clever effort manage to achieve high amplitude oscillations on a swing.