# Financial Time Series – Estimators for mean and autocovariance functions

Andreas Petersson

TMS088/MSA410 – March 2020

Mathematical Sciences, Chalmers University of Technology & University of Gothenburg, Sweden

# Mean and autocovariance function

**Definition**

Let $X$ be a *stationary* time series. The *mean* $\mu \in \mathbb{R}$ of $X$ is given by

$$\mu := \mathbb{E}(X_t)$$

for any $t \in \mathbb{Z}$. The *autocovariance function (ACVF)* $\gamma_X : \mathbb{Z} \to \mathbb{R}$ of $X$ is defined by

$$\gamma_X(h) := \mathsf{Cov}(X_{t+h}, X_t)$$

for $h \in \mathbb{Z}$ and $t \in \mathbb{Z}$.

Write $\gamma$ instead of $\gamma_X$ if it causes no confusion. Note that $\gamma$ is symmetric around zero.

# Estimators for mean and autocovariance functions

## Definition

Let $X = (X_t, t \in \mathbb{N})$ be a time series. The *sample mean* $\bar{X}_n$ of $X$ is given by

$$\bar{X}_n := n^{-1} \sum_{t=1}^{n} X_t.$$

The *sample autocovariance function* $\hat{\gamma}$ is defined by

$$\hat{\gamma}(h) := n^{-1} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X})$$

for $h = 0, \ldots, n-1$. Furthermore the *sample autocorrelation function* $\hat{\rho}$ is given by

$$\hat{\rho}(h) := \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

for $h = 0, \ldots, n-1$.

Extended to $\hat{\gamma}(h) := \hat{\gamma}(|h|)$ for $h < 0$.

# Convergence of sample mean

**Proposition**

*Let $X$ be a stationary time series with mean $\mu$ and autocovariance $\gamma_X$. Then*

$$\lim_{n\to\infty} \mathsf{Var}(\bar{X}_n) = \lim_{n\to\infty} \mathbb{E}\left((\bar{X}_n - \mu)^2\right) = 0$$

*if* $\sum_{|h|<\infty} |\gamma_X(|h|)| < +\infty$.

# Some facts about the sample mean and autocovariance

- Exercise: If $X$ is Gaussian then

$$n^{1/2}(\bar{X}_n - \mu) \sim \mathcal{N}\left(0, \sum_{|h|<n} (1 - n^{-1}|h|)\gamma(h)\right).$$

# Some facts about the sample mean and autocovariance

- Exercise: If $X$ is Gaussian then

$$n^{1/2}(\bar{X}_n - \mu) \sim \mathcal{N}\left(0, \sum_{|h|<n} (1 - n^{-1}|h|)\gamma(h)\right).$$

- Exercise: $\bar{X}_n$ is unbiased if $X$ is stationary, $\hat{\gamma}$ is biased (even if the factor $n^{-1}$ is replaced by $(n-h)^{-1}$).

# Some facts about the sample mean and autocovariance

- Exercise: If $X$ is Gaussian then

$$n^{1/2}(\bar{X}_n - \mu) \sim \mathcal{N}\left(0, \sum_{|h|<n} (1 - n^{-1}|h|)\gamma(h)\right).$$

- Exercise: $\bar{X}_n$ is unbiased if $X$ is stationary, $\hat{\gamma}$ is biased (even if the factor $n^{-1}$ is replaced by $(n-h)^{-1}$).
- Box & Jenkins: $\hat{\gamma}$ is reliable for $n \geq 50$, $h \leq n/4$.

## Some facts about the sample mean and autocovariance

- Exercise: If $X$ is Gaussian then

$$n^{1/2}(\bar{X}_n - \mu) \sim \mathcal{N}\left(0, \sum_{|h|<n} (1 - n^{-1}|h|)\gamma(h)\right).$$

- Exercise: $\bar{X}_n$ is unbiased if $X$ is stationary, $\hat{\gamma}$ is biased (even if the factor $n^{-1}$ is replaced by $(n-h)^{-1}$).
- Box & Jenkins: $\hat{\gamma}$ is reliable for $n \geq 50$, $h \leq n/4$.
- The $k$-dimensional *sample covariance matrix*:

$$\hat{\Gamma}_k := \begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(k-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \cdots & \hat{\gamma}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}(k-1) & \hat{\gamma}(k-2) & \cdots & \hat{\gamma}(0) \end{pmatrix}$$

## Example: Quarterly earnings of H&M

Let the set $(x_t)_{t=1}^{42}$ be the quarterly earnings of H&M. The sample mean is found to be $\bar{x} \approx 4007$.

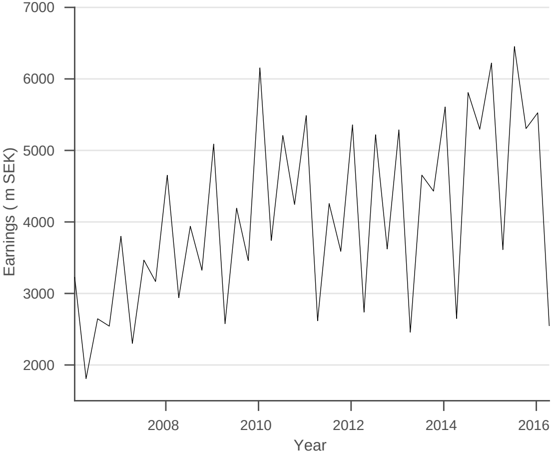


**Figure:** Quarterly earnings of H&M from January 2006 through April 2016.

# Example: Quarterly earnings of H&M

The sample autocorrelation function:

```
n=length(data);
mx=mean(data);
lags=10;
gamma=zeros(1,lags+1);
for h=0:lags
        gamma(h+1)=(data(1+h:end)-mx)*(data
            (1:end-h)-mx)'/n;
end
acf=gamma/gamma(1);
```

# Example: Quarterly earnings of H&M

The sample autocorrelation function:

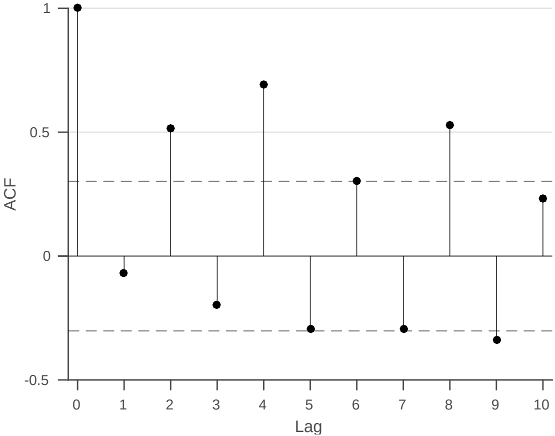


**Figure:** Sample autocorrelation function for the H&M data.

# Application: Testing for IID

If $Y = (Y_1, \ldots, Y_n) \sim \mathrm{IID}(\mu, \sigma^2)$ then $\hat{\rho}(h)$, $h = 1, 2, 3, \ldots$ are approximately iid and $\mathcal{N}(0, n^{-1})$ for large $n$ and small $h$.

# Application: Testing for IID

If $Y = (Y_1, \dots, Y_n) \sim \mathrm{IID}(\mu, \sigma^2)$ then $\hat{\rho}(h)$, $h = 1, 2, 3, \dots$ are approximately iid and $\mathcal{N}(0, n^{-1})$ for large $n$ and small $h$.
If $Y = (Y_1, \dots, Y_n) \sim \mathrm{IID}(\mu, \sigma^2)$ then the test statistic

$$\lambda := n \sum_{i=1}^{h} \hat{\rho}(i)^2.$$

is approximately $\chi_h^2$ distributed.

# Application: Testing for IID

If $Y = (Y_1, \dots, Y_n) \sim \mathrm{IID}(\mu, \sigma^2)$ then $\hat{\rho}(h)$, $h = 1, 2, 3, \dots$ are approximately iid and $\mathcal{N}(0, n^{-1})$ for large $n$ and small $h$.
If $Y = (Y_1, \dots, Y_n) \sim \mathrm{IID}(\mu, \sigma^2)$ then the test statistic

$$\lambda := n \sum_{i=1}^{h} \hat{\rho}(i)^2.$$

is approximately $\chi_h^2$ distributed.

# Application: Testing for IID

## Method (Portmanteau test, Box–Pierce test)

We have

$$\begin{aligned} H_0 :& Y \sim \mathrm{IID}(\mu, \sigma^2), \\ H_1 :& Y \nsim \mathrm{IID}(\mu, \sigma^2). \end{aligned}$$

and the test statistic

$$\lambda := n \sum_{i=1}^{h} \hat{\rho}(i)^2.$$

$H_0$ is rejected at level $\alpha \in (0,1)$ if $\lambda > \chi^2_{1-\alpha,h}$.

# Application: Testing for IID

**Method (Ljung–Box test)**

A modification of the Portmanteau test. Use the test statistic

$$\lambda := n(n+2)\sum_{i=1}^{h}\frac{\hat{\rho}(i)^2}{n-i}. \tag{1}$$

Same rejection regions as in the Portmanteau test.

Apply the Ljung–Box test with $h = 4$ and $\alpha = 0.05$ to the quarterly earnings of H&M $\Rightarrow \lambda = 38.87 \gg \chi^2_{0.95,4} = 9.49 \Rightarrow H_0$ is *rejected*.

# Application: Be careful when testing for IID

Let $x = (x_t)_{t=1}^{205}$ be monthly observations of the Australian Trade Weighted Index (ATWI), a weighted sum of exchange rates between the Australian dollar and other currencies from January 1978 to January 1995.

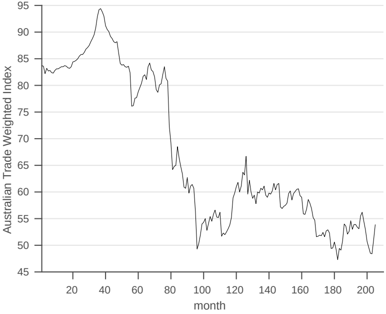


**Figure:** Monthly observations of the ATWI.

# Application: Be careful when testing for IID

Log-returns: $y = (y_t)_{t=1}^{204}$ with $y_t = \log(y_{t+1}) - \log(y_t)$

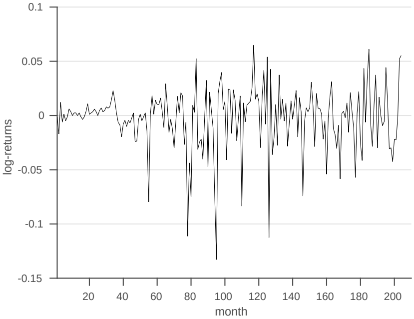


**Figure:** Log-returns of the ATWI.

# Application: Be careful when testing for IID

Sample ACF of log-returns:

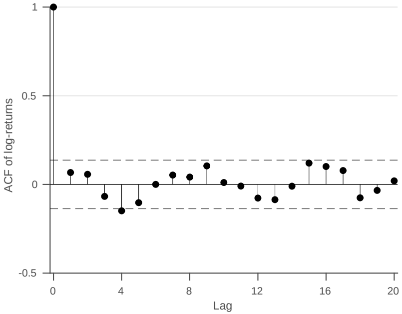


**Figure:** Sample ACF of log-returns.

Apply Ljung–Box test to $y$ with $h = 20$ and
$\alpha = 0.05 \Rightarrow \lambda = 24.37 < \chi^2_{0.95,20} = 31.41 \Rightarrow H_0$ is *not rejected*.

# Application: Be careful when testing for IID

Sample ACF of absolute log-returns:

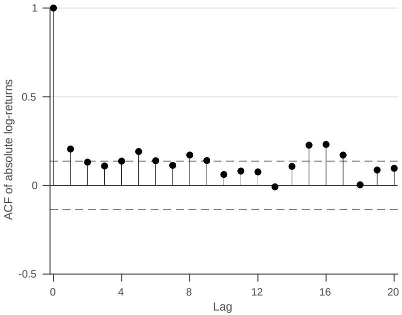


**Figure:** Sample ACF of absolute log-returns.

Apply Ljung–Box test to $|y|$ with $h = 20$ and
$\alpha = 0.05 \Rightarrow \lambda = 84.25 \gg \chi^2_{0.95,20} = 31.41 \Rightarrow H_0$ is *rejected*.