Financial Time Series – Linear time series

Andreas Petersson

TMS088/MSA410 – April 2020

UNIVERSITY OF
GOTHENBURG

Mathematical Sciences, Chalmers University of Technology & University of Gothenburg, Sweden

Linear processes

Definition

A stochastic process X is called a *linear process* if

$$X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j} \tag{1}$$

for all $t \in \mathbb{Z}$, where $Z \sim \mathrm{WN}(0, \sigma^2)$ and $(\psi_j, j \in \mathbb{Z})$ is a sequence of real numbers with $\sum_{j \in \mathbb{Z}} |\psi_j| < +\infty$.

Linear processes

Definition

A stochastic process X is called a *linear process* if

$$X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j} \tag{1}$$

for all $t \in \mathbb{Z}$, where $Z \sim \mathrm{WN}(0, \sigma^2)$ and $(\psi_j, j \in \mathbb{Z})$ is a sequence of real numbers with $\sum_{j \in \mathbb{Z}} |\psi_j| < +\infty$.

With $\psi(B) := \sum_{j \in \mathbb{Z}} \psi_j B^j$ we write (1) as $X_t = \psi(B) Z_t$

Existence and ACVF

Proposition

Let Y be a stationary time series with mean zero and ACVF γ_Y and let $\psi(B)$ be the filter from before. Then the time series X defined by

$$X_t := \psi(B)Y_t$$

for all $t \in \mathbb{Z}$ is stationary with mean zero and ACVF

$$\gamma_X(h) = \sum_{j,k\in\mathbb{Z}} \psi_j \psi_k \gamma_Y(h + j - k)$$

for all $h \in \mathbb{Z}$.

If $Y = Z \sim \mathrm{WN}(0, \sigma^2)$ then

$$\gamma_X(h) = \sum_{j\in\mathbb{Z}} \psi_j \psi_{h+j} \sigma^2$$

for all $h \in \mathbb{Z}$.

Existence and ACVF

Proof.

Existence and ACVF

Proof.

$$\mathbb{E}(|X_t|) \leq \sum_{j\in\mathbb{Z}} |\psi_j| \, \mathbb{E}(|Y_{t-j}|) \leq \left(\sum_{j\in\mathbb{Z}} |\psi_j| \right) \gamma_Y(0)^{1/2} < +\infty.$$

Existence and ACVF

Proof.

$$\mathbb{E}(|X_t|) \leq \sum_{j\in\mathbb{Z}} |\psi_j| \, \mathbb{E}(|Y_{t-j}|) \leq \left(\sum_{j\in\mathbb{Z}} |\psi_j|\right) \gamma_Y(0)^{1/2} < +\infty.$$

$$\mathbb{E}(X_t) = \mathbb{E}(\psi(B)Y_t) = \mathbb{E}\left(\sum_{j\in\mathbb{Z}} \psi_j Y_{t-j}\right)$$

□

Existence and ACVF

Proof.

$$\mathbb{E}(|X_t|) \leq \sum_{j\in\mathbb{Z}} |\psi_j| \, \mathbb{E}(|Y_{t-j}|) \leq \left(\sum_{j\in\mathbb{Z}} |\psi_j| \right) \gamma_Y(0)^{1/2} < +\infty.$$

$$\begin{aligned} \mathbb{E}(X_t) = \mathbb{E}(\psi(B)Y_t) &= \mathbb{E}\left(\sum_{j\in\mathbb{Z}} \psi_j Y_{t-j} \right) \\ &= \sum_{j\in\mathbb{Z}} \psi_j \, \mathbb{E}(Y_{t-j}) = 0, \end{aligned}$$

□

Existence and ACVF

Proof.

$$
\begin{aligned}
\mathsf{Cov}(X_t, X_{t+h}) &= \mathbb{E}\left(\sum_{j,k\in\mathbb{Z}} \psi_j Y_{t-j} \psi_k Y_{t+h-k}\right) \\
&= \sum_{j,k\in\mathbb{Z}} \psi_j \psi_k \, \mathbb{E}(Y_{t-j} Y_{t+h-k}) \\
&= \sum_{j,k\in\mathbb{Z}} \psi_j \psi_k \gamma_Y((t+h-k)-(t-j)) \\
&= \sum_{j,k\in\mathbb{Z}} \psi_j \psi_k \gamma_Y(h+j-k),
\end{aligned}
$$

$\square$