Financial Time Series – ARMA processes

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Linear time series

Definition

A stochastic process X is called a *linear process* if it has the representation

$$X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j}$$

for all $t \in \mathbb{Z}$, where $Z \sim \mathrm{WN}(0, \sigma^2)$ and $(\psi_j, j \in \mathbb{Z})$ is a sequence of real numbers with $\sum_{j \in \mathbb{Z}} |\psi_j| < +\infty$.

Definition

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A time series X is called an *autoregressive process of order* p or $\mathrm{AR}(p)$ *process* if X is stationary and if for all $t \in \mathbb{Z}$

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t,$$

where $Z \sim \mathrm{WN}(0, \sigma^2)$.

Definition

A time series X is called a *moving average process of order* q or $\mathrm{MA}(q)$ *process* if X is stationary and if for all $t \in \mathbb{Z}$

$$X_t = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where $Z \sim \mathrm{WN}(0, \sigma^2)$.

Definition

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A time series X is an $\mathrm{ARMA}(p, q)$ *process* if X is stationary and if for all $t \in \mathbb{Z}$

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},$$

where $Z \sim \mathrm{WN}(0, \sigma^2)$ and the polynomials $(1 - \sum_{j=1}^{p} \phi_j z^j)$ and $(1 + \sum_{j=1}^{q} \theta_j z^j)$ have no common zeros. Further a time series X is called an $\mathrm{ARMA}(p, q)$ *process with mean* μ if $X - \mu$ is an $\mathrm{ARMA}(p, q)$ process.

Existence and uniqueness

Proposition

A stationary solution X of the equation

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j},\, t \in \mathbb{Z},$$

exists and is the unique stationary solution if and only if

$$\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \neq 0$$

for all $z \in \mathbb{C}$ with $|z| = 1$.

Existence and uniqueness

Proof.

$\square$

Existence and uniqueness

Proof.

$$X_t := \phi(B)^{-1}\theta(B)Z_t = \phi(B)^{-1}\sum_{i=0}^{q}\theta_i Z_{t-i} = \sum_{j=-\infty}^{\infty}\sum_{i=0}^{q}\chi_j\theta_i Z_{t-i-j}$$

$\square$

Existence and uniqueness

Proof.

$$X_t := \phi(B)^{-1}\theta(B)Z_t = \phi(B)^{-1}\sum_{i=0}^{q}\theta_i Z_{t-i} = \sum_{j=-\infty}^{\infty}\sum_{i=0}^{q}\chi_j\theta_i Z_{t-i-j}$$

$$\sum_{k=0}^{p}\chi_{-k}\phi_k = -1 \text{ and } \sum_{k=0}^{p}\chi_{m-k}\phi_k = 0 \text{ for all } m \neq 0$$

□

Result from last lecture

Recall: $\psi(B) := \sum_{j\in\mathbb{Z}} \psi_j B^j$ and $\sum_{j\in\mathbb{Z}} |\psi_j| < +\infty$.

Proposition

Let Y be a stationary time series with mean zero and ACVF γ_Y. Then the time series X defined by

$$X_t := \psi(B)Y_t$$

for all $t \in \mathbb{Z}$ is stationary with mean zero and ACVF

$$\gamma_X(h) = \sum_{j,k\in\mathbb{Z}} \psi_j \psi_k \gamma_Y(h + j - k)$$

for all $h \in \mathbb{Z}$.

If $Y = Z \sim \mathrm{WN}(0, \sigma^2)$ then for all $h \in \mathbb{Z}$,

$$\gamma_X(h) = \sum_{j\in\mathbb{Z}} \psi_j \psi_{h+j} \sigma^2.$$

An ARMA process as a linear time series

An index change shows that

$$X_t = \phi(B)^{-1}\theta(B)Z_t = \sum_{k=-\infty}^{\infty}\left(\sum_{i=0}^{q}\chi_{k-i}\theta_i\right)Z_{t-k}.$$

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Set $\psi(z) := \phi(z)^{-1}\theta(z)$.

$$X_t = \psi(B)Z_t = \sum_{k=-\infty}^{\infty}\psi_k Z_{t-k},$$

where $\psi_k = (\sum_{i=0}^{q}\chi_{k-i}\theta_i)$ are the coefficients in the Laurent series expansion of $\psi(z)$.

Causal ARMA processes

Definition

An $\mathrm{ARMA}(p, q)$ process X is *causal* or a *causal function of Z* if there exists a real-valued sequence $(\psi_j, j \in \mathbb{N}_0)$ such that $\sum_{j=0}^{\infty} |\psi_j| < +\infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

for all $t \in \mathbb{Z}$, i.e., if X is a moving average/$\mathrm{MA}(\infty)$ process.

Causal ARMA processes

Lemma

An $\mathrm{ARMA}(p,q)$ *process* X *is causal if and only if*

$$1 - \sum_{j=1}^{p} \phi_j z^j \neq 0$$

for all $z \in \mathbb{C}$ *with* $|z| \leq 1$*.*

Corollary

A causal $\mathrm{ARMA}(p,q)$ *process has a unique stationary solution.*

Causal ARMA processes

Proof.

□

Causal ARMA processes

For a causal process, $\psi(z)\phi(z) = \theta(z)$, or

$$(\psi_0 + \psi_1 z + \cdots)(1 - \phi_1 z - \cdots - \phi_p z^p) = (1 + \theta_1 z + \cdots + \theta_q z^q).$$

Causal ARMA processes

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$$(\psi_0 + \psi_1 z + \cdots)(1 - \phi_1 z - \cdots - \phi_p z^p) = (1 + \theta_1 z + \cdots + \theta_q z^q).$$

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j$$

for all $j \in \mathbb{N}_0$, where $\theta_0 := 1$, $\theta_j := 0$ for $j > q$, and $\psi_j := 0$ for $j < 0$.

Invertible ARMA processes

Definition

An $\mathrm{ARMA}(p,q)$ process X is *invertible* if there exists a real-valued sequence $(\pi_j, j \in \mathbb{N}_0)$ such that $\sum_{j=0}^{\infty} |\pi_j| < +\infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

for all $t \in \mathbb{Z}$, i.e., if X is an autoregressive/$\mathrm{AR}(\infty)$ process.

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An $\mathrm{ARMA}(p,q)$ *process* X *is invertible if and only if*

$$1 + \sum_{j=1}^{q} \theta_j z^j \neq 0$$

for all $z \in \mathbb{C}$ *with* $|z| \leq 1$.

Invertible ARMA processes

Definition

An $\mathrm{ARMA}(p,q)$ process X is *invertible* if there exists a real-valued sequence $(\pi_j, j \in \mathbb{N}_0)$ such that $\sum_{j=0}^{\infty} |\pi_j| < +\infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

for all $t \in \mathbb{Z}$, i.e., if X is an autoregressive/$\mathrm{AR}(\infty)$ process.

The sequence $(\pi_j, j \in \mathbb{N}_0)$ is determined by

$$\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} = -\phi_j$$

for $j \in \mathbb{N}_0$, where we set $\phi_0 := -1$, $\phi_j := 0$ for $j > p$, and $\pi_j := 0$ for $j < 0$.

If X is a non-causal and/or non-invertible ARMA process, then it is a causal/invertible ARMA process with respect to another white noise series!

An example

Example

Let $X := (X_t, t \in \mathbb{Z})$ be given by

$$X_t := Y_t(Z_t + Z_{t-1})$$

where $Z := (Z_t, t \in \mathbb{Z})$ is $\mathrm{IID}(0, \sigma_Z^2)$ and $Y := (Y_t, t \in \mathbb{Z})$ is a stationary time series independent of Z.

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$$\gamma_X(h) = \begin{cases} 2(\gamma_Y(h) + \mu_Y^2)\sigma_Z^2 & \text{if } h = 0, \\ (\gamma_Y(h) + \mu_Y^2)\sigma_Z^2 & \text{if } |h| = 1, \\ 0 & \text{else.} \end{cases}$$

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$$\gamma_X(h) = \begin{cases} 2\sigma_Z^2 & \text{if } h = 0, \\ \frac{\sigma_Z^2}{2} & \text{if } |h| = 1, \\ 0 & \text{else.} \end{cases}$$

An example

Represent X as an $\mathrm{MA}(1)$ process by

$$X_t = V_t + \theta_1 V_{t-1},\, V \sim \mathrm{WN}(0, \sigma_V^2).$$

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Note: $\gamma_X(0) = (1 + \theta_1^2)\sigma_V^2$ and $\gamma_X(1) = \theta_1 \sigma_V^2$

An example

Represent X as the $\mathrm{MA}(1)$ process

$$X_t = V_t + (2-\sqrt{3})V_{t-1},\, V \sim \mathrm{WN}(0, \frac{\sigma_Z^2}{2(2-\sqrt{3})}).$$