Financial Time Series – (P)ACF for ARMA processes

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ARMA processes

- $\phi(B)X_t = \theta(B)Z_t$
- $Z \sim \mathrm{WN}(0, \sigma^2)$
- $\phi(z) := 1 - \sum_{i=1}^{p} \phi_i z^i$
- $\theta(z) := 1 + \sum_{j=1}^{q} \theta_j z^j$

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- *Assume for the next two slides*: X is causal, i.e., for $(\psi_j, j \in \mathbb{N}_0)$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

- For all $j \in \mathbb{N}_0$, where $\theta_0 := 1$, $\theta_j := 0$ for $j > q$, and $\psi_j := 0$ for $j < 0$:

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j \tag{1}$$

Calculate ACVF of ARMA process

Method

Multiply the white noise expansions of X_{t+h} and X_t to get

$$\gamma(h) = \mathbb{E}(X_{t+h}X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}.$$

Calculate ACVF of ARMA process

Method

Multiply each side of the equations

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}$$

by X_{t-k} for $k \in \mathbb{N}_0$ and take expectations on each side.

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by X_{t-k} for $k \in \mathbb{N}_0$ and take expectations on each side. We obtain

$$\gamma(k) - \sum_{j=1}^{p} \phi_j \gamma(k-j) = \sigma^2 \sum_{j=0}^{q-k} \theta_{k+j} \psi_j \tag{2}$$

for $0 \leq k \leq q$ and

$$\gamma(k) - \sum_{j=1}^{p} \phi_j \gamma(k-j) = 0 \tag{3}$$

for $k > q$, where $\psi_j := 0$ for $j < 0$, $\theta_0 := 1$, and $\theta_j := 0$ for $j \notin \{0, \ldots, q\}$.

The PACF

Definition

For $X \sim \mathrm{ARMA}(p, q)$. The *partial autocorrelation function* α (*PACF*):

$$\alpha(0) := 1,$$
$$\alpha(h) := \phi_{hh}$$

for $h \geq 1$, where ϕ_{hh} is the last component of

$$\phi_h = \left((\gamma(i-j))_{i,j=1}^{h}\right)^{-1} (\gamma(1), \gamma(2), \ldots, \gamma(h))'.$$

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For $(x_1, \ldots, x_n)$, the *sample PACF* $\hat{\alpha}$ is given by

$$\hat{\alpha}(0) := 1,$$
$$\hat{\alpha}(h) := \hat{\phi}_{hh}$$

for $h \geq 1$, where $\hat{\phi}_{hh}$ is the last component of

$$\hat{\phi}_h = \left((\hat{\gamma}(i-j))_{i,j=1}^{h}\right)^{-1} (\hat{\gamma}(1), \hat{\gamma}(2), \ldots, \hat{\gamma}(h))'.$$

The PACF

- It can be shown that:

$$\alpha(1) = \mathsf{Cor}(X_{t+1}, X_t) = \rho(1)$$

and for $h \geq 2$

$$\alpha(h) = \mathsf{Cor}(X_{t+h} - b_{t+h}^l(X^h), X_t - b_t^l(X^h))$$

where $X^h := (X_{t+1}, \ldots, X_{t+h-1})$.

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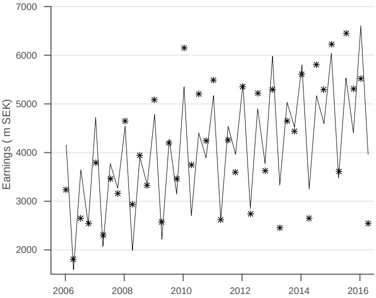
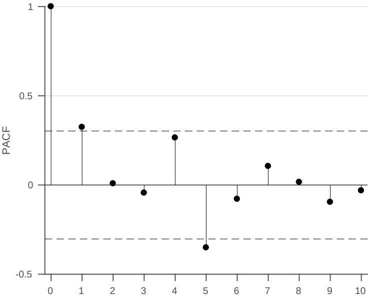
- Exercise: The PACF of a causal $\mathrm{AR}(p)$ process is zero for lags greater than p
- For causal $\mathrm{AR}(p)$ and large n: $(\hat{\alpha}(h))_{h=p+1}^{\infty} \sim \mathrm{IID}\,\mathcal{N}(0, n^{-1})$ approximately

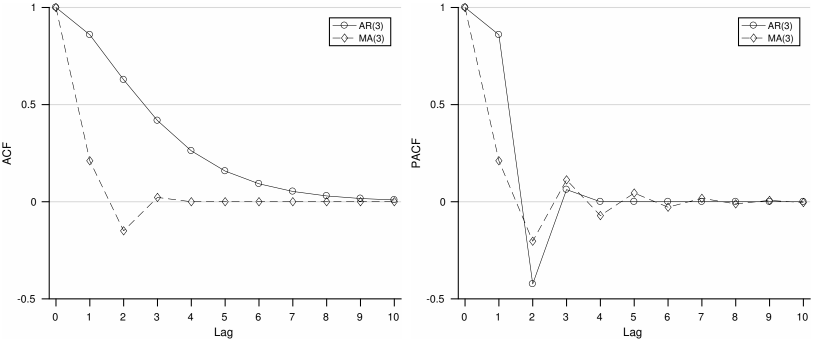
Let the set $(x_t)_{t=1}^{42}$ be the quarterly earnings of H&M.



(a) The H&M data with trend and seasonality



(b) Sample PACF for the *detrended and deseasonlized* H&M data.



(a) The ACF

(b) The PACF

Causal $\mathrm{AR}(3)$:

$$X_t - \frac{5}{4}X_{t-1} + \frac{1}{2}X_{t-2} - \frac{1}{16}X_{t-3} = Z_t,$$

and invertible $\mathrm{MA}(3)$:

$$X_t = Z_t + \frac{5}{4}Z_{t-1} - \frac{1}{2}Z_{t-2} + \frac{1}{16}Z_{t-3}.$$