Financial Time Series – Forecasting algorithms

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Two algorithms for recursive forecasting

- In this lecture, let $X^n := (X_1, \dots, X_n)$ for all $n \in \mathbb{N}$.
- Idea: Use $b^l_{n+1}(X^n)$ to compute $b^l_{n+2}(X^{n+1})$
- Let
$$\hat{X}_n := \begin{cases} 0 & \text{for } n = 1, \\ b^l_n(X^{n-1}) & \text{for } n > 1, \end{cases}$$
and
$$v_n := \mathrm{MSE}(\hat{X}_{n+1}, X_{n+1}) = \mathbb{E}((\hat{X}_{n+1} - X_{n+1})^2).$$
- $\hat{X}_{n+1} = \sum_{i=1}^n a_{ni} X_{n+1-i}$ where $a_{ni} := a_i$ in terms of the previous notation

The Durbin–Levinson algorithm

- Requirement :$\Gamma_n = (\gamma(i-j))_{i,j=1}^{n}$ non-singular for every n

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Method (Durbin–Levinson algorithm)

Compute the coefficients $a_{n1}, \ldots, a_{nn}$ recursively from the equations

$$a_{nn} := \left(\gamma(n) - \sum_{i=1}^{n-1} a_{(n-1)i} \gamma(n-i) \right) v_{n-1}^{-1},$$

$$\begin{pmatrix} a_{n1} \\ \vdots \\ a_{n(n-1)} \end{pmatrix} := \begin{pmatrix} a_{(n-1)1} \\ \vdots \\ a_{(n-1)(n-1)} \end{pmatrix} - a_{nn} \begin{pmatrix} a_{(n-1)(n-1)} \\ \vdots \\ a_{(n-1)1} \end{pmatrix},$$

and

$$v_n := v_{n-1}(1 - a_{nn}^2),$$

where $a_{11} = \gamma(1)/\gamma(0)$ and $v_0 := \gamma(0)$.

Innovations

- Setting: $X = (X_t, t \in \mathbb{Z})$ is a *not necessarily stationary* time series with $\mathbb{E}(X_t) = 0$, Var$(X_t^2) = \mathbb{E}(X_t^2) < \infty$ for all $t \in \mathbb{Z}$ and covariance function

$$\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) = \kappa(i, j)$$

- Let

$$\hat{X}_n := \begin{cases} 0 & \text{for } n = 1, \\ b_n^l(X^{n-1}) & \text{for } n > 1, \end{cases}$$

and

$$v_n := \mathrm{MSE}(\hat{X}_{n+1}, X_{n+1}) = \mathbb{E}((\hat{X}_{n+1} - X_{n+1})^2).$$

Lemma

There exist unique coefficients $(\theta_{ij},\, 1 \le j \le i \le n)$ *such that the best linear predictors satisfy*

$$\hat{X}_{n+1} = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & \text{for } n \ge 1. \end{cases}$$

Proof.

$\square$

Innovations

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Proof.

Suffices to show:

$$\hat{X}_{n+1} = \sum_{j=1}^{n} \theta_{n(n+1-j)}(X_j - \hat{X}_j).$$

We know that

$$\hat{X}_{n+1} = \sum_{j=1}^{n} a_{n(n+1-j)} X_j.$$

□

Innovations

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Proof.

Assume

$$\hat{X}_{j+1} = \sum_{i=1}^{j} \theta_{j(j+1-i)}(X_i - \hat{X}_i).$$

for all $j = 1, \ldots, n-1$. We know that

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Innovations

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There exist unique coefficients $(\theta_{ij}$*,* $1 \leq j \leq i \leq n)$ *such that the best linear predictors satisfy*

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Proof.

Assume

$$\hat{X}_{j+1} = \sum_{i=1}^{j} \theta_{j(j+1-i)}(X_i - \hat{X}_i).$$

for all $j = 1, \ldots, n-1$. We use that

$$\hat{X}_{n+1} = \sum_{j=1}^{n} a_{n(n+1-j)}(X_j - \hat{X}_j) + \sum_{j=1}^{n} a_{n(n+1-j)}\hat{X}_j$$

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Innovations

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Proof.

$\sum_{j=1}^{n} a_{n(n+1-j)} \hat{X}_j$

□

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Proof.

$$\sum_{j=1}^{n} a_{n(n+1-j)} \hat{X}_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{j-1} a_{n(n+1-j)} \theta_{(j-1)(j-i)} \Big) (X_i - \hat{X}_i)$$

$\square$

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Proof.

$$\begin{aligned} \sum_{j=1}^{n} a_{n(n+1-j)} \hat{X}_j &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{j-1} a_{n(n+1-j)} \theta_{(j-1)(j-i)} \Big) (X_i - \hat{X}_i) \\ &= \sum_{i=1}^{n-1} \Big(\sum_{j=i+1}^{n} a_{n(n+1-j)} \theta_{(j-1)(j-i)} \Big) (X_i - \hat{X}_i) \end{aligned}$$

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Proof.

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Proof.

$$\hat{X}_{n+1} = \sum_{j=1}^{n} \Big(a_{n(n+1-j)} + \sum_{i=j+1}^{n} a_{n(n+1-i)} \theta_{(i-1)(i-j)} \Big) (X_j - \hat{X}_j),$$

$\square$

Innovations algorithm

Method (Innovations algorithm)

Compute the coefficients $\theta_{n1}, \ldots, \theta_{nn}$ recursively from the equations

$$v_0 := \kappa(1,1)$$

and

$$\theta_{n(n-k)} := v_k^{-1} \left(\kappa(n+1, k+1) - \sum_{j=0}^{k-1} \theta_{k(k-j)} \theta_{n(n-j)} v_j \right)$$

for $0 \leq k < n$ and

$$v_n := \kappa(n+1, n+1) - \sum_{j=0}^{n-1} \theta_{n(n-j)}^2 v_j.$$

Solve for θ and v in the order $v_0, \theta_{11}, v_1, \theta_{22}, \theta_{21}, v_2, \theta_{33}, \theta_{32}, \theta_{31}, v_3, \ldots$

Innovations algorithm

Proof.

$$
\begin{aligned}
&\mathbb{E}(\hat{X}_{n+1}(X_{k+1} - \hat{X}_{k+1})) \\
&\quad = \mathbb{E}\left(\sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j})(X_{k+1} - \hat{X}_{k+1})\right) \\
&\quad = \sum_{j=1}^{n} \theta_{nj}\, \mathbb{E}((X_{n+1-j} - \hat{X}_{n+1-j})(X_{k+1} - \hat{X}_{k+1}))
\end{aligned}
$$

□

Innovations algorithm

Proof.

$$\mathbb{E}(X_{n+1}(X_{k+1} - \hat{X}_{k+1})) = \theta_{n(n-k)} v_k,$$

$\square$

Innovations algorithm

Proof.

$$\theta_{n(n-k)} = v_k^{-1}\,\mathbb{E}(X_{n+1}(X_{k+1} - \hat{X}_{k+1}))$$

$\square$

Innovations algorithm

Proof.

$$
\begin{aligned}
\theta_{n(n-k)} &= v_k^{-1}\,\mathbb{E}(X_{n+1}(X_{k+1} - \hat{X}_{k+1})) \\
&= v_k^{-1}\Big(\,\mathbb{E}(X_{n+1}X_{k+1}) - \sum_{i=1}^{k} \theta_{ki}\,\mathbb{E}(X_{n+1}(X_{k+1-i} - \hat{X}_{k+1-i}))\Big)
\end{aligned}
$$

□

Innovations algorithm

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&= v_k^{-1}\Big(\kappa(n+1, k+1) - \sum_{i=1}^{k} \theta_{ki}\theta_{n(n-(k-i))}v_{k-i}\Big)
\end{aligned}
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Innovations algorithm

Proof.

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\begin{aligned}
\theta_{n(n-k)} &= v_k^{-1}\,\mathbb{E}(X_{n+1}(X_{k+1} - \hat{X}_{k+1})) \\
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&= v_k^{-1}\Big(\kappa(n+1,k+1) - \sum_{i=1}^{k} \theta_{ki}\theta_{n(n-(k-i))}v_{k-i}\Big) \\
&= v_k^{-1}\Big(\kappa(n+1,k+1) - \sum_{j=0}^{k-1} \theta_{k(k-j)}\theta_{n(n-j)}v_j\Big)
\end{aligned}
$$

$\square$

Innovations algorithm

Proof.

$$
\begin{aligned}
&\mathbb{E}(X_{n+1}\hat{X}_{n+1}) \\
&\quad = \mathbb{E}(\hat{X}_{n+1}^2) + \mathbb{E}((X_{n+1} - \hat{X}_{n+1})\hat{X}_{n+1})
\end{aligned}
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&\quad = \mathbb{E}(\hat{X}_{n+1}^2) + \sum_{j=1}^{n} \theta_{nj}\, \mathbb{E}((X_{n+1} - \hat{X}_{n+1})(X_{n+1-j} - \hat{X}_{n+1-j}))
\end{aligned}
$$

□

Innovations algorithm

Proof.

□