Financial Time Series – Extensions of GARCH models

Andreas Petersson

TMS088/MSA410 – May 2020

Mathematical Sciences, Chalmers University of Technology & University of Gothenburg, Sweden

Random variance models

Definition

A stochastic process $X = (X_t, t \in \mathbb{Z})$ is said to follow a *random variance model* if

$$X_t = \sigma_t Z_t \tag{1}$$

for all $t \in \mathbb{Z}$, where $Z = (Z_t, t \in \mathbb{Z})$ is $\mathrm{IID}(0, 1)$ and $\sigma = (\sigma_t, t \in \mathbb{Z})$ is an unspecified stochastic process called the *volatility*. If X_t can be written as a deterministic function of $(Z_s, s \leq t)$ for all $t \in \mathbb{Z}$, then X is said to be *causal*.

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The *realized volatility*:

$$\hat{\sigma}_t^2 := (\tau - 1)^{-1} \sum_{j=t-\tau}^{t} (x_j - \bar{x}_t)^2$$

for observed data $(x_1, \ldots, x_n)$, fixed $\tau < n$, and $\tau < t \leq n$, where

$$\bar{x}_t := \tau^{-1} \sum_{j=t-\tau}^{t} x_j.$$

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A stochastic process $X = (X_t, t \in \mathbb{Z})$ is called a $\mathrm{GARCH}(p, q)$ *process* if it is a stationary solution to the GARCH equations

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where $Z \sim \mathrm{IID}(0, 1)$,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2,$$

with $\alpha_0 > 0$, $\alpha_j \geq 0$ for $j = 1, \ldots, p$, $\beta_i \geq 0$ for $i = 1, \ldots, q$.

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Usually $Z_t \sim \mathcal{N}(0, 1)$ or $\sqrt{\nu/(\nu - 2)} Z_t \sim t_\nu$ for all $t \in \mathbb{Z}$. The factor $\sqrt{\nu/(\nu - 2)}$ yields $\mathsf{Var}(Z_t) = 1$, Z_t follows a *generalized* or *non-standardized* t-distribution.

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where $Z \sim \mathrm{IID}(0, 1)$ has a symmetric distribution, i.e., Z_t and $-Z_t$ have the same distribution,

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p} \alpha_j g(Z_{t-j}) + \sum_{i=1}^{q} \beta_i \ln(\sigma_{t-i}^2),$$

where $g(x) = x + \lambda(|x| - \mathbb{E}(|Z_t|))$ and $\alpha_0, \alpha_1, \dots \alpha_p, \beta_1, \dots, \beta_q, \lambda$ are real numbers.

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$$\sigma_t^2 = \exp(\alpha_0) \times \prod_{j=1}^{p} \exp(\alpha_j g(Z_{t-j})) \times \prod_{i=1}^{q} \sigma_{t-i}^{2\beta_i}.$$

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- $\mathbb{E}(Z_t|Z_t|) = \mathbb{E}(-Z_t|-Z_t|) \implies \mathbb{E}(Z_t|Z_t|) = -\mathbb{E}(Z_t|Z_t|)$, i.e., that $\mathbb{E}(Z_t|Z_t|) = 0$ for all $t \in \mathbb{Z}$

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- Exercise: show that $g(Z) = (g(Z_t), t \in \mathbb{Z}) \sim \mathrm{WN}(0, 1 + \lambda^2 \,\mathsf{Var}(|Z_t|))$ and hence that $\ln(\sigma^2)/\alpha_{p'} := (\ln(\sigma_t^2)/\alpha_{p'}, t \in \mathbb{Z})$ is an $\mathrm{ARMA}(q, p - p')$ (where p' is the first $j \in \mathbb{N}$ such that $\alpha_j \neq 0$) process with mean $\mu = \alpha_0/(\alpha_{p'}(1 - \beta(1)))$.

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- Parameter estimation can be done via conditional MLE

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- Recall:

$$\phi(B)X_t^2 = \alpha_0 + \theta(B)\eta_t,$$

  where $\phi(z) = 1 - \alpha(z) - \beta(z)$, $\theta(z) = 1 - \beta(z)$ and $\eta_t = X_t^2 - \sigma_t^2$ is white noise

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- Idea: Let X^2 behave like an ARIMA process
- $\phi(B) = (1 - B)\tilde{\phi}(B)$, where $\tilde{\phi}$ is some polynomial such that $\tilde{\phi}(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1 \implies \phi$ has a simple root at 1

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where $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are non-negative numbers, $\alpha_0 > 0$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1$.

- A causal strictly stationary solution exists if the distribution of Z_t has unbounded support and not atom at zero (true if $Z \sim \mathrm{IID}\, \mathcal{N}(0, 1)$)
- Infinite (unconditional) variance!