Financial Time Series – Stationary process

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What is stationarity?

- Informal definition: A time series $X = (X_t, t \in \mathbb{Z})$ that "looks the same" everywhere – for all $n \in \mathbb{N}$ the distribution of $(X_t, X_{t+1}, \ldots, X_{t+n})$ is invariant w.r.t. $t \in \mathbb{Z}$

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- Replaces the notion of independent samples in "standard" statistics

Quarterly earnings of H&M

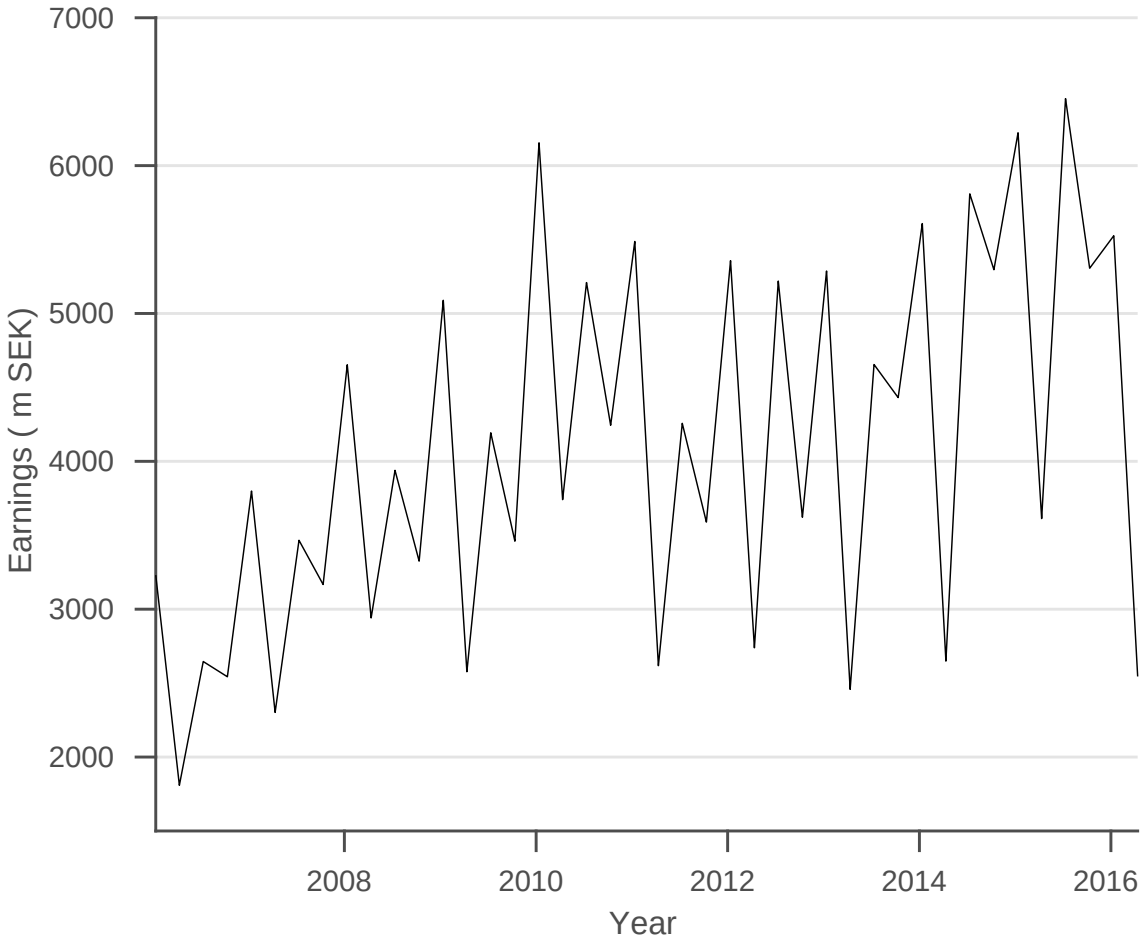

Figure: Quarterly earnings of H&M from January 2006 through April 2016.

Log-returns

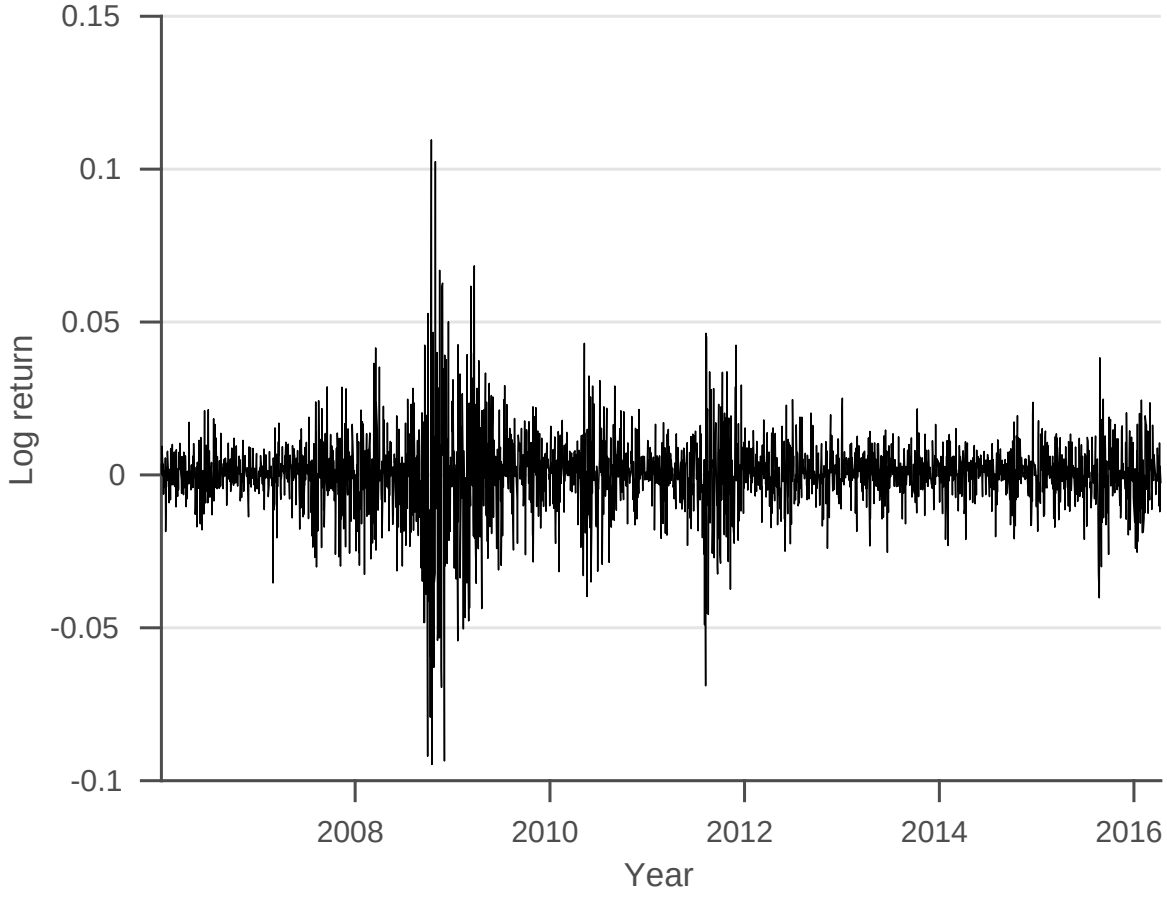

Figure: Daily log-returns of the S&P500 index from January 2006 to April 2016.

Stationarity continued

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- Whether a time series can be modeled as stationary depends on time frame and prior knowledge
- Time series data "is stationary" if there is a stationary time series model that reproduces its behaviour

Mean and covariance functions

Definition

Let $X = (X_t, t \in \mathbb{Z})$ be a stochastic process with $\mathsf{Var}(X_t) < +\infty$ for all $t \in \mathbb{Z}$. The *mean function* $\mu_X : \mathbb{Z} \to \mathbb{R}$ of X is given by

$$\mu_X(t) := \mathbb{E}(X_t)$$

for all $t \in \mathbb{Z}$ and the *covariance function* $\gamma_X : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is defined by

$$\gamma_X(r, s) := \mathsf{Cov}(X_r, X_s) = \mathbb{E}\left((X_r - \mu_X(r))(X_s - \mu_X(s))\right)$$

for all $r, s \in \mathbb{Z}$.

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1. there exists $\mu \in \mathbb{R}$ such that $\mu_X(t) = \mu$ for all $t \in \mathbb{Z}$ and
2. $\gamma_X(r, s) = \gamma_X(r + h, s + h)$ for all $r, s, h \in \mathbb{Z}$.

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Definition

A time series $X = (X_t, t \in \mathbb{Z})$ (for which $\mathrm{Var}(X_t) < \infty$ for all $t \in \mathbb{Z}$ is not necessarily true) is said to be *strictly stationary* if $(X_1, \ldots, X_n) \stackrel{d}{=} (X_{1+h}, \ldots, X_{n+h})$ for all $h \in \mathbb{Z}$ and $n \in \mathbb{N}$.

An example

Example

Let $X := (X_t, t \in \mathbb{Z})$ be given by

$$X_t := Y_t(Z_t + Z_{t-1})$$

where $Z := (Z_t, t \in \mathbb{Z})$ is $\mathrm{IID}(0, \sigma_Z^2)$ and $Y := (Y_t, t \in \mathbb{Z})$ is a stationary time series independent of Z. Show that X is a (weakly) stationary time series.

Autocovariance function

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Autocovariance function:

$$\gamma_X(h) := \gamma_X(h, 0) = \gamma_X(s + h, s)$$

for $h, s \in \mathbb{Z}$, for stationary time series.

Autocovariance function

Definition

Let X be a stationary time series. The *autocovariance function (ACVF)* $\gamma_X : \mathbb{Z} \to \mathbb{R}$ of X is defined by

$$\gamma_X(h) := \mathsf{Cov}(X_{t+h}, X_t)$$

for $h \in \mathbb{Z}$ and $t \in \mathbb{Z}$. The *autocorrelation function (ACF)* $\rho_X : \mathbb{Z} \to [-1, 1]$ of X is defined by

$$\rho_X(h) := \frac{\gamma_X(h)}{\gamma_X(0)}$$

for $h \in \mathbb{Z}$.

White noise

Definition

A stochastic process $X = (X_t, t \in \mathbb{Z})$ is called a *white noise* with mean μ and variance σ^2 if it is a stationary process with $\mathbb{E}(X_t) = \mu$, $t \in \mathbb{Z}$, and for $h \in \mathbb{Z}$

$$\gamma_X(h) = \begin{cases} \sigma^2 & \text{if } h = 0, \\ 0 & \text{else.} \end{cases}$$

If X is a white noise it is denoted by $X \sim \mathrm{WN}(\mu, \sigma^2)$.

Estimators for mean and autocovariance functions

Definition

Let $X = (X_t, t \in \mathbb{N})$ be a time series. The *sample mean* $\bar{X}_n$ of X is given by

$$\bar{X}_n := n^{-1} \sum_{t=1}^{n} X_t.$$

The *sample autocovariance function* $\hat{\gamma}$ is defined by

$$\hat{\gamma}(h) := n^{-1} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X})$$

for $h = 0, \ldots, n-1$. Furthermore the *sample autocorrelation function* $\hat{\rho}$ is given by

$$\hat{\rho}(h) := \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

for $h = 0, \ldots, n-1$.

Extended to $\hat{\gamma}(h) := \hat{\gamma}(|h|)$ for $h < 0$.