CHALMERS, GÖTEBORGS UNIVERSITET

SOLUTIONS FOR EXAM for
ARTIFICIAL NEURAL NETWORKS
October 25, 2021

COURSE CODES: **FFR 135, FIM 720 GU, PhD**

Maximum score on this exam: 12 points.
Maximum score for homework problems: 12 points.
To pass the course it is necessary to score at least 5 points on this written exam.
CTH >13.5 passed; >17 grade 4; >21.5 grade 5,
GU >13.5 grade G; > 19.5 grade VG.

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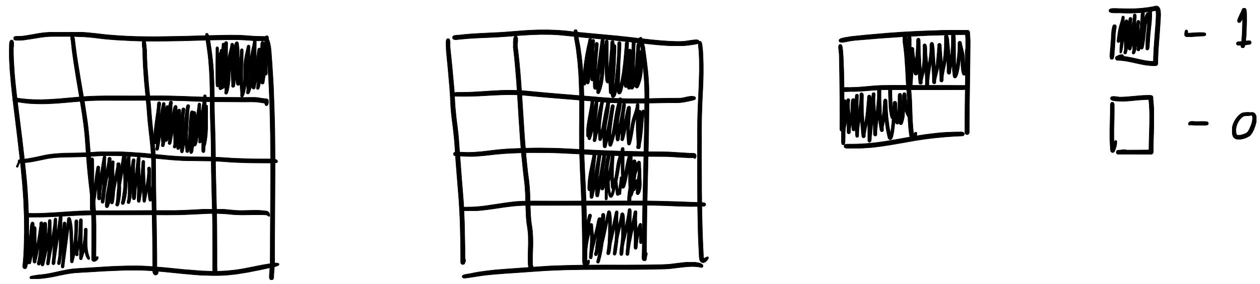
1. Convolutional network.

Convolutional network

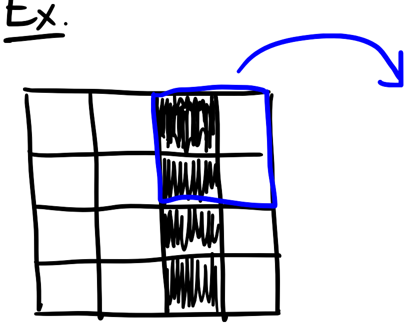
Arbitrary choice

Pattern 1 Pattern 2 Kernel



- Apply kernel to patterns with stride (1, 1) and padding (0, 0, 0, 0), using a ReLU activation function

Ex.



$$\begin{pmatrix} 1\cdot 0 & 0\cdot 1 \\ 1\cdot 1 & 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Sum the entries of the resulting matrix and apply ReLU activation function: $g(0+0+1+0) = 1$

- Resulting convolution layers:

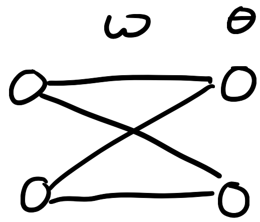
$$V^{(1)} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad V^{(2)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

- Apply (2×3) max-pooling layer with stride (1, 1)

$$M^{(1)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad M^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Fully connected classification layer with signum activation function sgn:

Two inputs from max-pooling layer and two output neurons



ω : (2×2) weight matrix

θ : (2×1) threshold vector

$$O_i^{(\mu)} = \mathrm{sgn}\left(\sum_j^2 \omega_{ij} M^{(\mu)} - \theta_i\right), \quad \mu = \text{pattern}$$

Pattern 1: $\begin{pmatrix} O_1^{(1)} \\ O_2^{(1)} \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(2\omega_{11} + 2\omega_{12} - \theta_1) \\ \mathrm{sgn}(2\omega_{21} + 2\omega_{22} - \theta_2) \end{pmatrix}$

Pattern 2: $\begin{pmatrix} O_1^{(2)} \\ O_2^{(2)} \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(\omega_{11} + \omega_{12} - \theta_1) \\ \mathrm{sgn}(\omega_{21} + \omega_{22} - \theta_2) \end{pmatrix}$

Choose: $\omega = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and $\theta = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

Pattern 1: $\begin{pmatrix} O_1^{(1)} \\ O_2^{(1)} \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(4 - 3) \\ \mathrm{sgn}(-4 + 3) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Pattern 2: $\begin{pmatrix} O_1^{(2)} \\ O_2^{(2)} \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(2 - 3) \\ \mathrm{sgn}(-2 + 3) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The patterns can be classified using the parameters

$$\omega = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } \theta = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

2. Boltzmann machine (a) Start with the KL divergence,

$$
\begin{aligned}
D_{KL} &= \sum_{\mu=1}^{p} P_{data}(x^\mu) \log \frac{P_{data}(x^\mu)}{P_B(s = x^\mu)} && (1) \\
&= -\sum_{\mu=1}^{p} P_{data}(x^\mu) \log \frac{P_B(s = x^\mu)}{P_{data}(x^\mu)}. && (2)
\end{aligned}
$$

Use the inequality $\log z \leq z - 1$, where the equality holds iff $z = 1$.

$$
\begin{aligned}
-\sum_{\mu=1}^{p} P_{data}(x^\mu) \log \frac{P_B(s = x^\mu)}{P_{data}(x^\mu)} &\geq -\sum_{\mu=1}^{p} P_{data}(x^\mu) \left[\frac{P_B(s = x^\mu)}{P_{data}(x^\mu)} - 1 \right], && (3) \\
&\geq -\sum_{\mu=1}^{p} \left[P_B(s = x^\mu) - P_{data}(x^\mu) \right], && (4)
\end{aligned}
$$

Since the probabilities P_B, P_{data} must sum to 1,

$$
-\sum_{\mu=1}^{p} P_{data}(x^\mu) \log \frac{P_B(s = x^\mu)}{P_{data}(x^\mu)} \geq -\left[1 - 1\right] \geq 0, \tag{5}
$$

with the equality valid if and only if $P_B(s = x^\mu) = P_{data}(x^\mu)$.
(b) Hidden units are required because 3-point correlations must be considered to differentiate between bars and stripes.

3. Linearly inseparable classification problem The weights and thresholds for the three neurons can be inferred by writing the equations of the three decision boundaries:

$$f_1(x_1, x_2) = -x_1 - x_2 + 2 = 0 \tag{6}$$
$$f_2(x_1, x_2) = x_1 + 0\,x_2 + 2 = 0 \tag{7}$$
$$f_3(x_1, x_2) = 0\,x_1 + x_2 + 2 = 0. \tag{8}$$

For each decision boundary, $f_i(x_1, x_2) = 0$ on the boundary, $f_i(x_1, x_2) > 0$ on the side containing the origin, $(0, 0)$, and $f_i(x_1, x_2) < 0$ on the other side of the decision boundary. Since $f_i(0, 0) > 0$ for all i, the sign of the coefficients of x_1, x_2 are correct.
Thus,

$$w = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \theta = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} \tag{9}$$

Finally, choosing $W = [1, 1, 1]$ and $\Theta = 5/2$ maps the region enclosed by the three decision boundaries to $+1$ but the region outside to -1.

4. Backpropagation

Backpropagation

(a) With $H = \frac{1}{2}(t - V^{(L)})^2$ and $\delta w^{(L,L-1)} = -\eta \frac{\partial H}{\partial w^{(L,L-1)}}$

$$\frac{\partial H}{\partial w^{(L,L-1)}} = \frac{1}{2}\frac{\partial}{\partial w^{(L,L-1)}}\left(t - V^{(L)}\right)^2 = -(t - V^{(L)})\frac{\partial V^{(L)}}{\partial w^{(L,L-1)}}$$

$$= -(t - V^{(L)})\frac{\partial}{\partial w^{(L,L-1)}} g(b^{(L)})$$

$$(*) = -(t - V^{(L)})\, g'(b^{(L)}) \frac{\partial}{\partial w^{(L,L-1)}}\left(w^{(L,L-1)}V^{(L-1)} + w^{(L,L-2)}V^{(L-2)} - \theta^{(L)}\right)$$

$$= -(t - V^{(L)})\, g'(b^{(L)})\, V^{(L-1)}$$

$$\boxed{\therefore \quad \delta w^{(L,L-1)} = \eta (t - V^{(L)})\, g'(b^{(L)})\, V^{(L-1)}}$$

(b) Performing the same steps up until $(*)$ we have for $\delta w^{(L-1,L-2)}$:

$$\frac{\partial H}{\partial w^{(L-1,L-2)}} = -(t - V^{(L)})\, g'(b^{(L)})\, w^{(L,L-1)} \frac{\partial V^{(L-1)}}{\partial w^{(L-1,L-2)}}$$

$$= -(t - V^{(L)})\, g'(b^{(L)})\, w^{(L,L-1)} g'(b^{(L-1)})\, V^{(L-2)}$$

$$\boxed{\therefore \quad \delta w^{(L-1,L-2)} = \eta (t - V^{(L)})\, g'(b^{(L)})\, w^{(L,L-1)} g'(b^{(L-1)})\, V^{(L-2)}}$$

For $\delta w^{(L-2,L-3)}$ we have:

$$\frac{\partial H}{\partial w^{(L-2,L-3)}} = -(t - V^{(L)})\, g'(b^{(L)}) \frac{\partial}{\partial w^{(L-2,L-3)}}\left(w^{(L,L-1)}V^{(L-1)} + w^{(L,L-2)}V^{(L-2)} - \theta^{(L)}\right)$$

$$= -(t - V^{(L)})\, g'(b^{(L)}) \left(\omega^{(L,L-1)} \frac{\partial V^{(L-1)}}{\partial \omega^{(L-2,L-3)}} + \omega^{(L,L-2)} \frac{\partial V^{(L-2)}}{\partial \omega^{(L-2,L-3)}} \right)$$

- $$\frac{\partial V^{(L-1)}}{\partial w^{(L-2,L-3)}} = g'(b^{(L-1)})\, \omega^{(L-1,L-2)} \frac{\partial V^{(L-2)}}{\partial \omega^{(L-2,L-3)}}$$

  $$= g'(b^{(L-1)})\, \omega^{(L-1,L-2)}\, g'(b^{(L-2)})\, V^{(L-3)}$$

- $$\frac{\partial V^{(L-2)}}{\partial w^{(L-2,L-3)}} = g'(b^{(L-2)})\, V^{(L-3)}$$

Thus we have:

$$\boxed{\begin{aligned} &\therefore\ \delta w^{(L-2,L-3)} \\ &= -(t - V^{(L)})\, g'(b^{(L)}) \Big(\omega^{(L,L-1)} g'(b^{(L-1)})\, \omega^{(L-1,L-2)} g'(b^{(L-2)})\, V^{(L-3)} \\ &\qquad + \omega^{(L,L-2)} g'(b^{(L-2)})\, V^{(L-3)} \Big) \end{aligned}}$$

5. Binary stochastic neuron

(a) Assuming only neuron m was updated, $s_m \to s'_m$ while the other neurons remained in the same state: $s_i \to s'_i = s_i \forall i \neq m$, let us start by writing the energy H:

$$H = -\frac{1}{2}\Big(\sum_{i\neq m, j\neq m} w_{ij}s_i s_j + \sum_{i\neq m} w_{im}s_i s_m + \sum_{j\neq m} w_{mj}s_m s_j + w_{mm}s_m s_m\Big)$$
$$+ \sum_{i\neq m}\theta_i s_i + \theta_m s_m.$$

Now we use the symmetery of the weights, $w_{mj} = w_{jm}$, and that $w_{mm} = 0$,

$$H = -\frac{1}{2}\Big(\sum_{i\neq m, j\neq m} w_{ij}s_i s_j + 2\sum_{j\neq m} w_{mj}s_m s_j\Big) + \sum_{i\neq m}\theta_i s_i + \theta_m s_m. \tag{10}$$

Similarly, the updated energy H' is,

$$H' = -\frac{1}{2}\Big(\sum_{i\neq m, j\neq m} w_{ij}s_i s_j + \sum_{i\neq m} w_{im}s_i s'_m + \sum_{j\neq m} w_{mj}s'_m s_j + w_{mm}s'_m s'_m\Big)$$
$$+ \sum_{i\neq m}\theta_i s_i + \theta_m s'_m.$$

where we have used the fact that $s_i \to s'_i = s_i \forall i \neq m$. Now simpify using symmetry of weights and vanishing diagonals,

$$H' = -\frac{1}{2}\Big(\sum_{i\neq m, j\neq m} w_{ij}s_i s_j + 2\sum_{j\neq m} w_{mj}s'_m s_j\Big) + \sum_{i\neq m}\theta_i s_i + \theta_m s'_m. \tag{11}$$

Subtracting Eq. (10) from (11),

$$\Delta H = -(s'_m - s_m)\Big(\sum_{j\neq m} w_{mj}s_j - \theta_m\Big) = -b_m(s'_m - s_m). \tag{12}$$

where $w_{mm} = 0$ is used again in the last equality to write $\sum_{j\neq m} w_{mj}s_j - \theta_m = \sum_j w_{mj}s_j - \theta_m = b_m$.

(b) Here one needs to consider different cases and show that Equation (3) in the exam is always equivalent to Equation (4a) in the exam.

Case 1: $s'_m = 1, s_m = -1$

Equation (4a) gives:

$$P(-1 \to 1) = \frac{1}{1 + e^{\beta \Delta H_m}} = \frac{1}{1 + e^{-2\beta b_m}}$$

.

Equation (3) gives: $s'_m = 1$ with probability

$$p(b_m) = \frac{1}{1 + e^{-2\beta b_m}}$$

.
Case 2: $s'_m = -1, s_m = -1$.
Equation (4a): Use conservation of probability, $P(-1 \to 1) + P(-1 \to -1) = 1 \implies P(-1 \to -1) = 1 - P(-1 \to 1)$,

$$P(-1 \to -1) = 1 - \frac{1}{1 + e^{-2\beta b_m}} = \frac{1}{1 + e^{2\beta b_m}}$$

.
Equation (3) gives: $s'_m = -1$ with probability

$$1 - p(b_m) = 1 - \frac{1}{1 + e^{-2\beta b_m}} = \frac{1}{1 + e^{2\beta b_m}}$$

. **Case 3:** $s'_m = -1, s_m = 1$
Equation (4a) gives:

$$P(1 \to -1) = \frac{1}{1 + e^{\beta \Delta H_m}} = \frac{1}{1 + e^{2\beta b_m}}$$

.
Equation (3) gives: $s'_m = -1$ with probability

$$1 - p(b_m) = \frac{1}{1 + e^{2\beta b_m}}$$

. **Case 4:** $s'_m = 1, s_m = 1$ Equation (4a): Use conservation of probability, $P(1 \to -1) + P(1 \to 1) = 1 \implies P(1 \to 1) = 1 - P(1 \to -1)$,

$$P(1 \to 1) = 1 - \frac{1}{1 + e^{2\beta b_m}} = \frac{1}{1 + e^{-2\beta b_m}}$$

.
Equation (3) gives: $s'_m = 1$ with probability

$$p(b_m) = \frac{1}{1 + e^{-2\beta b_m}}$$

.
Thus, we have shown that in all 4 possible cases, the two update rules are equivalent.

6. Oja's rule
(a) We start with the given learning rule:

$$\begin{aligned}\delta \boldsymbol{w} &= \eta y(\boldsymbol{x} - y\boldsymbol{w}),\\ &= \eta(\boldsymbol{x}y - y^2\boldsymbol{w}),\\ &= \eta[\boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{w} - (\boldsymbol{w}^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{w})\boldsymbol{w}],\end{aligned}$$

Where for the first time we have written $y = \boldsymbol{w}^\top \boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{w}$, while for the second term: $y^2 = yy = \boldsymbol{w}^\top \boldsymbol{x}\boldsymbol{x}^\top \boldsymbol{w}$. Now avergaing $\delta\boldsymbol{w}$ over the data distribution,

$$\langle \delta \boldsymbol{w}\rangle = \eta[\langle \boldsymbol{x}\boldsymbol{x}^\top\rangle \boldsymbol{w} - (\boldsymbol{w}^\top \langle \boldsymbol{x}\boldsymbol{x}^\top\rangle \boldsymbol{w})\boldsymbol{w}].$$

Let $\mathbb{C} \equiv \langle \boldsymbol{x}\boldsymbol{x}^\top\rangle$, then the above equation reads,

$$\langle \delta \boldsymbol{w}\rangle = \eta[\mathbb{C}\boldsymbol{w} - (\boldsymbol{w}^\top \mathbb{C}\boldsymbol{w})\boldsymbol{w}].$$

Assume that $\boldsymbol{w} = \boldsymbol{w}^*$ is the normalized maximal eigenvector of the matrix $\mathbb{C}$. That is, $\mathbb{C}\boldsymbol{w}^* = \lambda_1 \boldsymbol{w}^*$ where $\boldsymbol{w}^{*\top}\boldsymbol{w} = 1$ and λ_1 is the maximal eigenvalue. We obtain,

$$\begin{aligned}\langle \delta \boldsymbol{w}\rangle &= \eta[\mathbb{C}\boldsymbol{w}^* - (\boldsymbol{w}^{*\top}\mathbb{C}\boldsymbol{w}^*)\boldsymbol{w}^*],\\ &= \eta[\lambda_1\boldsymbol{w}^* - \lambda_1(\boldsymbol{w}^{*\top}\boldsymbol{w}^*)\boldsymbol{w}^*],\\ &= \eta[\lambda_1\boldsymbol{w}^* - \lambda_1\boldsymbol{w}^*],\\ &= 0.\end{aligned}$$

Thus we have shown that the normalized maximal eigenvector $\boldsymbol{w}^*$ of $\mathbb{C}$ is a steady state of the given learning rule.