# TMS165/MSA350 Stochastic Calculus
# Written exam Friday 19 August 2022 8.30–12.30

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Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

**Task 1.** Find the quadratic variation process of the Itô process $\{\int_0^t \mathrm{e}^{B(s)^2} ds\}_{t\geq 0}$.

**(5 points)**

**Task 2.** Show that for a two-dimensional continuous random variable $(X, Y)$ it holds that $\mathbf{E}\{X|\sigma(Y)\} = \int_{-\infty}^{\infty} x \, \frac{f_{X,Y}(x,Y)}{f_Y(Y)} \, dx$. (Hint: For an $A \in \sigma(Y)$ we have $\mathbf{1}_A(\omega) = \mathbf{1}_B(Y(\omega))$ for $\omega \in \Omega$ for some $B \subseteq \mathbb{R}$.) **(5 points)**

**Task 3.** Let $\{W(t)\}_{t\geq 0}$ be a Wiener process with drift given by $W(t) = \sigma \, B(t) + \mu \, t$ for some constants $\sigma, \mu \in \mathbb{R}$. Find the conditional probability density function of $W(t)$ given that $W(s) = x$ for $0 < s < t$. **(5 points)**

**Task 4.** Consider a solution $X(t)$ to a (not necessarily time-homogeneous) SDE with generator $L_t$. Show that if $f(x,t)$ solves the backward PDE $L_t f(x,t) + \frac{\partial f}{\partial t}(x,t) = 0$ for $t \in [0,T]$ with $f(x,T) = g(x)$ then $f(x,t) = \mathbf{E}\{g(X(T))|X(t) = x\}$. **(5 points)**

**Task 5.** Consider a solution $\{X(t)\}_{t\geq 0}$ to the Langevin SDE $dX(t) = -\alpha \, X(t) \, dt + \sigma \, dB(t)$. How can an observation of $X(t)$ be used to estimate the parameters $\alpha, \sigma > 0$ if their values are not known? **(5 points)**

**Task 6.** Explain what is Itô-Taylor expansion. **(5 points)**

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# Solutions to Written Exam August 2022

**Task 1.** As the process under consideration is both continuous and FV (as increasing) it has zero quadratic variation.

**Task 2.** $\int_{-\infty}^{\infty} x \, \frac{f_{X,Y}(x,Y)}{f_Y(Y)} \, dx$ is $\sigma(Y)$-measurable and for $A \in \sigma(Y)$ with $\mathbf{1}_A = \mathbf{1}_B(Y)$ for $B \subseteq \mathbb{R}$ we have $\mathbf{E}\big\{\mathbf{1}_A \int_{-\infty}^{\infty} x \, \frac{f_{X,Y}(x,Y)}{f_Y(Y)} \, dx\big\} = \int_{-\infty}^{\infty} \mathbf{1}_B(y) \int_{-\infty}^{\infty} x \, f_{X,Y}(x,y) \, dxdy = \mathbf{E}\{\mathbf{1}_A X\}$.

**Task 3.** We have $(W(t)|W(s)=x) = (\sigma B(t)+\mu t|\sigma B(s)+\mu s = x) = \sigma\,(B(t)-B(s)) + \mu\, t + (\sigma B(s)|\sigma B(s) = x-\mu s)$ so that $(W(t)|W(s)=x)$ is $\mathrm{N}(x+\mu(t-s), \sigma^2(t-s))$-distributed as $B(t)-B(s)$ and $B(s)$ are independent.

**Task 4.** This is the proof of Theorem 6.6 in Klebaner's book.

**Task 5.** It is explained on the middle of page 286 in Klebaner's book how to estimate $\alpha$. As for $\sigma$, the solution $X(t) = \mathrm{e}^{-\alpha t}\big(X(0) + \int_0^t \sigma\, \mathrm{e}^{\alpha s} dB(s)\big)$ has quadratic variation process $[X,X](t) = \mathrm{e}^{-2\alpha t}\sigma^2 \int_0^t \mathrm{e}^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha t})$ so that we may, e.g., take $\hat{\sigma}^2 = 2\,\hat{\alpha}\,[X,X](t)/(1-\mathrm{e}^{-2\hat{\alpha} t})$ for some suitable $t$.

**Task 6.** See Stig Larsson's lecture notes.