Class Lectures (for Chapter 3)

Motivation

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- For all sets $A \subseteq R$ and $x \in R$,

$$\ell(A + x) = \ell(A).$$

$A + x = \{a + x : a \in A\}$.

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The construction of Lebesgue measure will be quite a bit of work.

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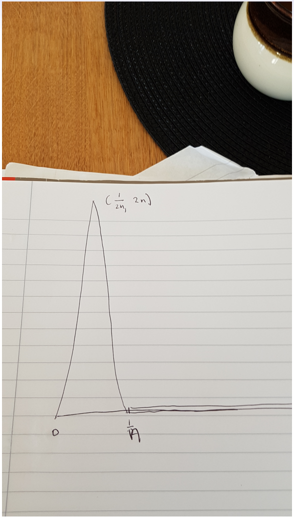
Question: If f_n is a nonnegative continuous function on $[0,1]$ for each n and if

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $x \in [0,1]$ (we say f_n goes to 0 pointwise in this case), does it follow that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0?$$



Question: When *will* we be able to conclude from the fact that f_n goes to 0 pointwise that the integrals converge to 0?

Algebras and σ-algebras

Algebras and σ-algebras

Definition

Let X be a nonempty set. An **algebra** or **field** of subsets of X is a collection $\mathcal{A}$ of subsets of X which is "closed under finite set theoretic operations"; i.e.

(1). $X \in \mathcal{A}$, $\emptyset \in \mathcal{A}$

(2). $A_1, A_2, \ldots, A_n$ each in $\mathcal{A}$ implies that $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ ($\mathcal{A}$ is closed under **finite** unions)

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Definition

Let X be a nonempty set. A σ-**algebra** or σ-**field** of subsets of X is a collection $\mathcal{M}$ of subsets of X which is an algebra and in addition (2) above is replaced by the stronger
(2'). $A_1, A_2, \ldots$ each in $\mathcal{M}$ implies that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ ($\mathcal{M}$ is closed under *countable* unions)

Generating σ-algebras

Proposition: Given a collection $\mathcal{E}$ of subsets of X (i.e., a subset of $\mathcal{P}(X)$), there is a smallest σ-algebra containing $\mathcal{E}$, denoted by $\sigma(\mathcal{E})$, called the σ-algebra generated by $\mathcal{E}$.

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This is clearly the smallest σ-algebra containing $\mathcal{E}$ since it is, by construction, contained inside of every σ-algebra which contains $\mathcal{E}$.
QED

The Borel sets

Recall the definition of an open set in R: O is open if for all $x \in O$, there exists $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq O$.

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Most sets (and very likely all sets) that you have seen are Borel sets.

Two further classes of sets

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Let X be a nonempty set.
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Definition

Let X be a nonempty set.
A nonempty collection $\mathcal{D}$ of subsets of X is called a $\mathcal{D}$-system if
a. $X \in \mathcal{D}$
b. $E, F \in \mathcal{D}$ and $E \subseteq F$ imply $F \backslash E (= F \cap E^c) \in \mathcal{D}$
and
c. $E_1 \subseteq E_2 \subseteq E_3, \ldots$ and $E_i \in \mathcal{D}$ for all i imply $\bigcup_i E_i \in \mathcal{D}$.

Dynkins $\pi - \lambda$ theorem

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Given a collection of $\mathcal{E}$ of subsets of X, we have previous defined $\sigma(\mathcal{E})$ as the smallest σ-algebra containing $\mathcal{E}$. We do something similar here.

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(Theorem 3.9 in JJ, Dynkin's $\pi - \lambda$ Theorem)
If $\mathcal{I}$ is a π-system, then

$$\mathcal{D}(\mathcal{I}) = \sigma(\mathcal{I}) .$$

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1. $m(\emptyset) = 0$
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Definition

A measure space $(X, \mathcal{M}, m)$ is a measurable space $(X, \mathcal{M})$ together with a measure m on it.

Easy example

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Example. Let $X = \{1, 2, 3, \dots\}$ and consider a vector $p_1, p_2, \dots$ of nonnegative numbers with $\sum_{i=1}^{\infty} p_i = 1$. Then let $\mathcal{M}$ be all subsets of X and for $S \subseteq X$, let

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We will get to more substantial examples soon, including Lebesgue measure.

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a. (Monotonicity) $E, F \in \mathcal{M}$, $E \subseteq F$ implies $m(E) \leq m(F)$.
b. (Continuity from below) $E_1 \subseteq E_2 \subseteq E_3, \ldots$ with each $E_i \in \mathcal{M}$ implies that

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d. (Continuity from above) $m(E_1) < \infty$ and $E_1 \supseteq E_2 \supseteq E_3, \dots$ implies

$$m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$$

Proof of a.

Using finite additivity in the first step and $m \geq 0$ in second step gives

$$m(F) = m(E) + m(F \backslash E) \geq m(E).$$

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We then have, using countable and finite additivity

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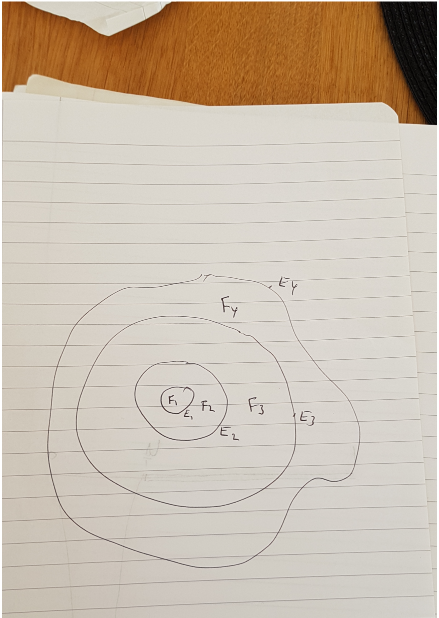
Picture for b.



Proof of c.

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We then have

$$m(\bigcup_{i=1}^\infty E_i) = m(\bigcup_{i=1}^\infty F_i) = \sum_{i=1}^\infty m(F_i) \leq \sum_{i=1}^\infty m(E_i).$$

Some properties measures may have

Definition

A measure space $(X, \mathcal{M}, m)$ is **complete** if (i) $B \in \mathcal{M}$, (ii) $m(B) = 0$ and (iii) $A \subseteq B$ imply that $A \in \mathcal{M}$ (which then of course implies that $m(A) = 0$).

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Given a measure space $(X, \mathcal{M}, m)$, a property (formally a subset of X) is said to occur **almost everywhere** abbreviated a.e. (**almost surely** abbreviated a.s. if one is doing probability theory) if the set of x's where the property fails is contained inside of a set of measure 0.

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A measure space $(X, \mathcal{M}, \mu)$ is called σ-**finite** if there exist subsets $A_1, A_2, \dots$ so that $X = \bigcup_i A_i$ and $\mu(A_i) < \infty$ for all i.

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Definition

Assume $(X, \mathcal{M}, \mu)$ is a measure space with all single points being measurable. An **atom** is a point x with $\mu(\{x\}) > 0$. $(X, \mathcal{M}, \mu)$ is called **atomic** if $\mu(\mathcal{A}^c) = 0$ where $\mathcal{A}$ is the set of atoms. $(X, \mathcal{M}, \mu)$ is called **continuous** if there is no atom.

Existence and construction of Lebesgue measure

Theorem

There exists a translation invariant measure m on $(R, \mathcal{B})$ such that $m([a, b]) = b - a$ for all $a < b$. (m will then be Lebesgue measure restricted to $\mathcal{B}$.)

Translation invariant means $m(A + x) = m(A)$ for all $A \in \mathcal{B}$ and $x \in R$.

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STEP 2: Using the notion of length for intervals in R, we construct *Lebesgue outer measure* which will be an outer measure (as will be defined in STEP 1). This will be defined for ALL subsets and should be viewed as the first attempt to construct Lebesgue measure. It will not be countably additive.

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$$\mu^\star(\bigcup_{j=1} A_j) \leq \sum_{i,j=1} |I_i^j| = \sum_{j=1}(\sum_{i=1} |I_i^j|) \leq \sum_{j=1}(\mu^\star(A_j)+\epsilon/2^j) = \sum_{j=1} \mu^\star(A_j)+\epsilon.$$

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Looking at the first and last term, since this inequality holds for all $\epsilon > 0$, we get

$$\mu^\star(\bigcup_{j=1} A_j) \leq \sum_{j=1} \mu^\star(A_j).$$

QED

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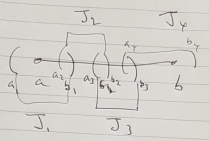
$\geq$ Assume $[a, b] \subseteq \bigcup_i I_i$. By compactness we can find an integer N so that $[a, b] \subseteq \bigcup_{i=1}^N I_i$. To complete the proof we need to show that

$$b - a \leq \sum_{i=1}^{N} |I_i|$$

which is very *believable* to say the least. See the picture for the proof.

STEP 3 Picture



$$\sum_{i=1}^{N} |I_i| \geq \sum_{i=1}^{k} |J_i|$$

$$= \sum_{i=1}^{k} (b_i - a_i)$$

$$= b_k - a_1 + \sum_{i=1}^{k-1} \underbrace{(b_i - a_{i+1})}_{\geq 0}$$

$$\geq b - a.$$

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Now we can state

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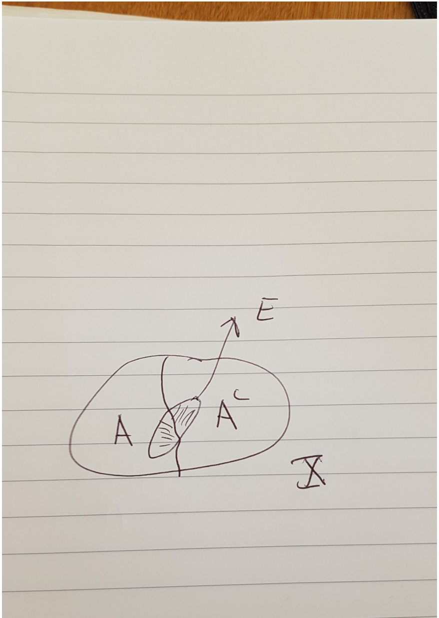

STEP 5

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Since $\mathcal{M}$ is a σ-algebra and $\mathcal{B}$ is the smallest σ-algebra containing the sets $(-\infty, a)$ and (b, ∞), it is enough to show that $(-\infty, a) \in \mathcal{M}$.

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Hence we obtain

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Since the LHS is $\geq$ the RHS for all coverings of E by open intervals, we can take the infimum of the LHS over all such coverings and obtain (1).

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Hence we can restrict $\mu^\star$ from $\mathcal{M}$ down to $\mathcal{B}$ obtaining the desired measure space $(R, \mathcal{B}, \mu^\star|_{\mathcal{B}})$.

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Finally, it is clear from the definition of the outer measure that $\mu^\star(A + x) = \mu^\star(A)$ for all sets A and $x \in R$. Hence $\mu^\star|_{\mathcal{B}}$ (as well as $\mu^\star|_{\mathcal{M}}$) is translation invariant.

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The proof will be broken into a number of steps.

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Fix $E \subseteq X$. Noting that

$$A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$$

and that this is a disjoint union, we have, using subadditivity,

$$\mu^\star(E \cap (A \cup B)) + \mu^\star(E \cap (A \cup B)^c) \leq$$

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$$A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$$

and that this is a disjoint union, we have, using subadditivity,

$$\mu^\star(E \cap (A \cup B)) + \mu^\star(E \cap (A \cup B)^c) \leq$$

$$\mu^\star(E \cap (A \cap B)) + \mu^\star(E \cap (A^c \cap B)) + \mu^\star(E \cap (A \cap B^c)) + \mu^\star(E \cap (A^c \cap B^c)).$$

Using measurability of A applied to $E \cap B$ for the sum of the first two terms and applied to $E \cap B^c$ for the sum of the second two terms, this equals

$$\mu^\star(E \cap B) + \mu^\star(E \cap B^c)$$

Proof of Caratheodory's Theorem

a. $\mathcal{M}$ **is an algebra.**
(i). $\emptyset \in \mathcal{M}$ is immediate.
(ii). $\mathcal{M}$ is closed under complementation since the definition is symmetric in A and A^c.
(iii). We need to show $A, B \in \mathcal{M}$ implies $A \cup B \in \mathcal{M}$.
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where the last equality follows from the measurability of B. Hence $A \cup B \in \mathcal{M}$.

Proof of Caratheodory's Theorem

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Now use induction. (Note that only one of the two sets was required to be measurable for this.)

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$$\mu^\star(E \cap B_n) = \mu^\star(E \cap B_n \cap A_n) + \mu^\star(E \cap B_n \cap A_n^c) = \mu^\star(E \cap A_n) + \mu^\star(E \cap B_{n-1}).$$

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This argument can be repeated inductively to obtain

$$\mu^\star(E \cap B_n) = \sum_{i=1}^{n} \mu^\star(E \cap A_i). \tag{2}$$

Proof of Caratheodory's Theorem

Now, using measurability of B_n together with (2), we have that for any n

$$\mu^\star(E) = \mu^\star(E \cap B_n) + \mu^\star(E \cap B_n^c) = \sum_{i=1}^{n} \mu^\star(E \cap A_i) + \mu^\star(E \cap B_n^c) \geq$$

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where we used subadditivity and the definition of B in the last inequality.

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where we used subadditivity and the definition of B in the last inequality. This establishes that $B \in \mathcal{M}$ and therefore that $\mathcal{M}$ is a σ-algebra .

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In particular, taking $E = B$, we obtain

$$\mu^\star(B) = \sum_{i=1}^{\infty} \mu^\star(A_i)$$

as desired.

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Hence if we have $B \in \mathcal{M}$, $\mu^\star(B) = 0$ and $A \subseteq B$, it follows that $\mu^\star(A) = 0$ and hence from the above $A \in \mathcal{M}$, as desired.

Two further classes of sets: REFRESHER

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Definition

Let X be a nonempty set.
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c. $E_1 \subseteq E_2 \subseteq E_3, \ldots$ and $E_i \in \mathcal{D}$ for all i imply $\bigcup_i E_i \in \mathcal{D}$.

Dynkins $\pi - \lambda$ theorem: REFRESHER

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Theorem

(Theorem 3.9 in JJ, Dynkin's $\pi - \lambda$ Theorem)
If $\mathcal{I}$ is a π-system, then

$$\mathcal{D}(\mathcal{I}) = \sigma(\mathcal{I}) .$$

Uniqueness of Lebesgue measure on the Borel sets

Uniqueness of Lebesgue measure on the Borel sets

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Assume that μ_1 and μ_2 are two measures on $(X, \ \sigma(\mathcal{I}) \ )$ such that

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Then $\mu_1 = \mu_2$.
Applying this to $X = [0, 1]$ and $\mathcal{I}$ being the set of open intervals implies that there is only one measure on $([0, 1], \mathcal{B}_{[0,1]})$ which agrees with "length" on intervals.

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Assume μ_1 and μ_2 are two such measures and let

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Our goal is to show that $D = \sigma(\mathcal{I})$. (Of course we have $\subseteq$.)

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Step 1: D is a $\mathcal{D}$-system. (Proof at end.)

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Step 3. Using Dynkin's $\pi - \lambda$ Theorem for the equality and steps 1 and 2 for the containment below, we have

$$\sigma(\mathcal{I}) = \mathcal{D}(\mathcal{I}) \subseteq \mathsf{D}$$

and hence $\mu_1 = \mu_2$.

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and hence $B \setminus A \in \mathrm{D}$.
c. If $E_1 \subseteq E_2 \subseteq E_3, \ldots$ and $E_i \in \mathrm{D}$ for all i, then using continuity from below for both measures, we have

$$\mu_1(\bigcup_i E_i) = \lim_{n \to \infty} \mu_1(E_n) = \lim_{n \to \infty} \mu_2(E_n) = \mu_2(\bigcup_i E_i)$$

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a,b, and c imply that D is a $\mathcal{D}$-system.
QED

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There does not exist a translation invariant measure on all subsets of R which gives length for intervals and hence there exist nonmeasurable sets.

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Assume μ is such a measure. Define an equivalence relation $\sim$ on $[0, 1]$ by

$$x \sim y \text{ if } x - y \in Q \ (Q \text{ denotes the rational numbers})$$

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Each equivalence class is countable and so the number of equivalence classes is uncountable. Let A consist of one element from each of the equivalence classes. What is $\mu(A)$?

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Subproof.

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To see this, if $x \in [0,1]$, choose $y \in A$ with $x - y \in Q$. Then $x = y + (x - y) \in A + (x - y)$. Since $x - y \in [-1,1] \cap Q$, we obtain (4).

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Now $\mu(A + q) = \mu(A)$ for all q by the assumed translation invariance.

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Clearly

$$\bigcup_{q \in [0,1] \cap Q} (A + q) \subseteq [0, 2] \tag{5}$$

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Each of the sets on the left hand side has measure $\mu(A)$ and hence if $\mu(A) > 0$, then the LHS would have infinite measure, contradicting the RHS has measure 2. Hence $\mu(A) = 0$.
QED

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Theorem

(Theorem 1.14 in F) If μ_0 is a premeasure on $(X, \mathcal{A})$, then there exists a measure μ on $(X, \sigma(\mathcal{A}))$ with $\mu(A) = \mu_0(A)$ for all $A \in \mathcal{A}$. If μ_0 is σ-finite on X, then μ is unique. (Uniqueness can fail in the non-σ-finite case.)

Distribution functions on $[0, 1]$

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Proposition: Let μ be a finite Borel measure on $[0, 1]$ and define $F : [0, 1] \to [0, \mu([0, 1])]$ by

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$$\lim_{s \downarrow t} F(s) = \lim_{n \to \infty} F(t + \frac{1}{n}) = \lim_{n \to \infty} \mu([0, t + \frac{1}{n}]) = \mu([0, t]) = F(t).$$

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Concerning left continuity, F jumps at the atoms of μ:

$$F(t) - \lim_{s \uparrow t} F(s) = \mu([0, t]) - \lim_{n \to \infty} \mu([0, t - \frac{1}{n}]) = \mu([0, t]) - \mu([0, t)) = \mu(\{t\}).$$

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Proposition: Let F be a nonnegative weakly increasing and right continuous function on $[0, 1]$ mapping into $[0, \infty)$. Then there exists a finite Borel measure μ on $[0, 1]$ satisfying

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Definition

The Cantor set, C, is defined to be $\bigcap_n C_n$.

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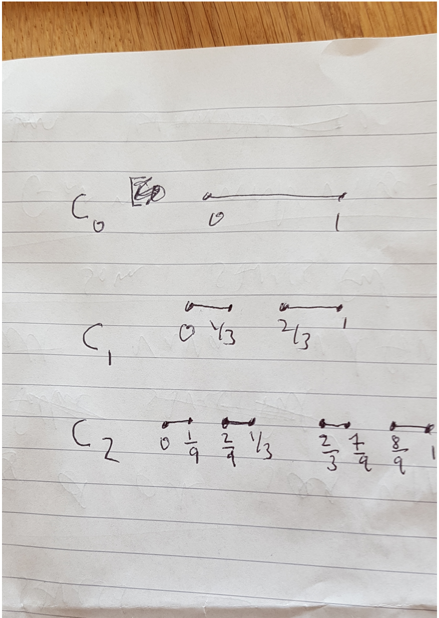

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The important feature of this measure is that it will have no atoms and it will give all of its weight to C, a set of Lebesgue measure 0. Such measures are called **continuous singular**.

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This is a crucial lemma in probability theory.

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Since this holds for each n and the RHS is the tail of a convergent series, we have that $m(\limsup E_i) = 0$.
QED