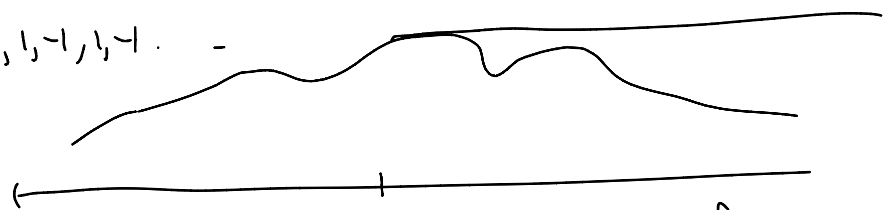
7 Folland 1.8

$\lim\limits_{n\to\infty} a_n$, $\overline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} \underbrace{\sup_{k\ge n} a_k}_{\text{dec in } n}$

1, -1, 1, -1, 1, -1 . . .



$\underline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} \underbrace{\inf_{k\ge n} a_k}_{\text{inc. in } n}$

$\underline{\lim}\, E_n = \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} E_k$

$= \{x : x \text{ is in all of the } E_n\text{'s except finitely many}\}$

$\mu(\underline{\lim}\, E_n) \le \underline{\lim\limits_{n\to\infty}}\, \mu(E_n)$

~~basic~~ baby version of Fatou's Lemma

Pf $\mu\left(\bigcap\limits_{k=n}^{\infty} E_k\right) \le \mu(E_j) \quad \forall j \ge n.$

$\Rightarrow \underbrace{\mu\left(\bigcap\limits_{k=n}^{\infty} E_k\right)}_{\text{inc in } n} \le \inf\limits_{j\ge n} \mu(E_j) \le \underline{\lim}\, \mu(E_j)$

$\mu(\underline{\lim}\, E_k) = \mu\left(\bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} E_k\right) \underset{\text{cont. from below}}{=} \lim\limits_{n\to\infty} \mu\left(\bigcap\limits_{k=n}^{\infty} E_k\right) \le \underline{\lim\limits_{n\to\infty}}\, \mu(E_n)$ $\square$

$\mu\left(\overline{\lim_{n\to\infty}} E_n\right) \geq \overline{\lim_{n\to\infty}}\, \mu(E_n)$ if

$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) < \infty \longrightarrow$?

Eg. $([0,\infty)$, Leb. measure$)$.

$E_n = [n, n+1]$ LHS 0 RHS $= 1$

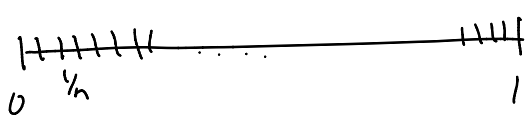
$\emptyset$

Hint: let $F_k = \bigcup_{j=k}^{\infty} E_j \setminus E_k$

$\mu(\underline{\lim}\, F_k) \leq \underline{\lim}\, \mu(F_k)$ by part 1.

Riemann intg.

$\int_0^1 f\,dx$

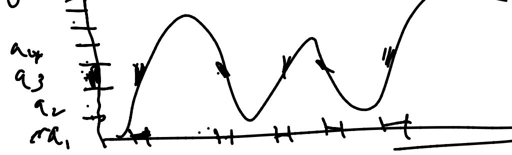


$\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n}$ let $n \to \infty$.

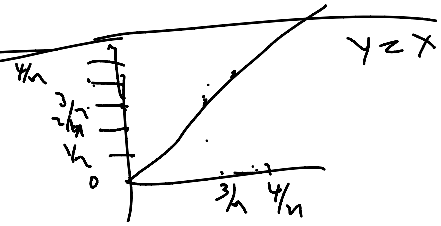
Lebesgue integral. Break the y-axis into small piece

$f(x) = I_Q(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$

not Riemann intg



$\sum_{i=1}^{n} m\{x : f(x) \in [a_2, a_3]\}\ a_3$



$\mathcal{A}$. pg 7 of Lecture notes:

Example of an alg. not a σ-alg.

$X = \mathbb{N}$, $\quad \mathcal{Q} = \{ A \subseteq \mathbb{N} : |A| < \infty$ or $\underbrace{|\mathbb{N} \setminus A| < \infty}_{A \text{ cofinite}} \}$

$\mathcal{Q}$ is a ~~σ~~ alg. not a σ-alg.

E = even integers. $E \notin \mathcal{Q}$ $\quad E = \bigcup_{i=1}^{\infty} \underset{\in \mathcal{Q}}{\{2i\}} \Rightarrow \mathcal{Q}$ not a σ-alg.

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pg 8. L.N. X set. $\quad \mathcal{F}_1 = \{X, \phi\}$, $\mathcal{F}_2 = \{$all ctable ~~and~~ or cocountable sets$\}$

$\mathcal{F}_3 = \cancel{\mathcal{L}} \ \mathcal{P}(X) = \{$all subsets of $X\}$.

$\mathcal{F}_1, \mathcal{F}_3$ trivially σ-alg. $\quad \mathcal{F}_2$ is a σ-alg.

Pf $\ni X, \phi$ ✓ $\quad A \in \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_2$. (symmetry)

$A_1, A_2, A_3 \dots \in \mathcal{F}_2$. NTS $\bigcup A_i \in \mathcal{F}_2$

case 1 all A_i's are ctble. $\Rightarrow \bigcup A_i$ ctble $\Rightarrow \bigcup A_i \in \mathcal{F}_2$.

case 2 at least one A_i is cocountable. A_n.

Then $\bigcup A_i$ cocountable since

$$|X \setminus \bigcup A_i| \le |X \setminus A_n| \qquad \square$$

$\mathcal{F}_1 = \mathcal{F}_2 \iff |X| = 1$.

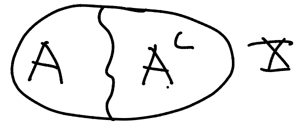
$\Leftarrow$ $|X| = 1$, only 1 σ-alg.

$\Rightarrow$ $|X| \ge 2$. choose $x \in X$. $\{x\} \in \mathcal{F}_2$ $\{x\} \notin \mathcal{F}_1$

$\mathcal{F}_2 = \mathcal{F}_3 \iff X$ ctble.

$\Leftarrow$ Easy $\quad \Rightarrow$. assume X not ctble.

WTS $\mathcal{F}_2 \ne \mathcal{F}_3$.



want to find $A \subseteq X$

with A unctble, A^c unctble. Such an $A \in \mathcal{F}_3$; $A \notin \mathcal{F}_2$

$X \times \{0\} = \{(x,0) : x \in X\}$. $\quad X \times \{1\} = \{(x,1) : x \in X\}$.

$|X \times \{0\}| = |X| = |X \times 1|$. $\quad (X \times \{0\}) \cup (X \times \{1\}) \longrightarrow X$

$\exists$ $\underline{f \text{ bij}}$ since $|X| = \infty$.

$A = f(X \times \{0\})$.

Folland 1.3.

$\mathcal{M}$ is an inf. σ-alg. (i.e. $\exists$ ∞ # of sets in $\mathcal{M}$).

(a) $\mathcal{M}$ contains a sequence of nonempty disjoint sets. $\underline{B_1, B_2, B_3 \ldots}$ disjoint nonempty

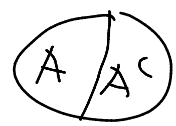
(b) $\text{card}(\mathcal{M}) \geq c$ (= card. of real #'s)

(a) call $A \in \mathcal{M}$ finite if

$$|\{E \in \mathcal{M} : E \subseteq A\}| < \infty \qquad \text{inf o.w.}$$

Note X is inf. by assumption. Choose $A \in \mathcal{M}$, $A \neq \emptyset$, $A \neq X$.

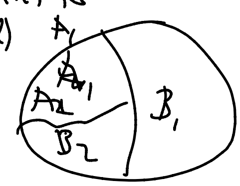
crucial observ. is either A or A^c is infinite



$$\Big(B \in \mathcal{M} \Rightarrow B = \underset{\in \mathcal{M}}{(B \cap A)} \cup \underset{\in \mathcal{M}}{(B \cap A^c)}\Big)$$

Let A_1 piece which is infinite, B_1 inf. or finite

A_1 inf. Break A_1 into A_2, B_2 partition (nontrivial) such that A_2 is inf.



Keep going $\Rightarrow$ $B_1, B_2, B_3 \ldots$ disjoint + nonempty.

(b) $c = |[0,1]| = |\{0,1\}^{\mathbb{N}}|$

$\{0,1\}^{\mathbb{N}}$ = infinite binary sequences

Find an $\underline{\text{injective}}$ map from $\{0,1\}^{\mathbb{N}}$ to $\mathcal{M}$

$\Rightarrow |\mathcal{M}| \geq |\{0,1\}^{\mathbb{N}}|$. done

$x \neq y \Rightarrow f(x) \neq f(y)$

$0110100100010000 \longrightarrow B_2 \cup B_3 \cup B_5 \cup B_8 \cup B_{12}$

1 2 3 4 5 6 7 8 9 10 11 12

f inj since B_i's are disjoint.

· PS 14 Lecture notes.

$(X, \mathfrak{m}, \mu)$ is σ-finite, then ~~#~~ $\mathcal{Q}$ (= atoms)
is at most ctble.

Pf. σ-finite $X = \bigcup_{i=1}^{\infty} A_i$ s.t. $\mu(A_i) < \infty$

If we can show $\forall i$ $|\mathcal{Q} \cap A_i|$ ctble, then done.

 NTS $\mu(B) < \infty$, then $B \cap \mathcal{Q}$ ctble

Pf. Let $F_K = \{ x \in B \cap \mathcal{Q} : \mu\{x\} \geq \frac{1}{K} \}$.

claim: $|F_K| < \infty$. Since if infinite,

$\mu(B) \geq \mu(F_K) \geq \frac{1}{K} \infty = \infty$. $\not\!\!$ ↯

~~Also~~ $B \cap \mathcal{Q} = \bigcup_{K=1}^{\infty} F_K$. $\Rightarrow$ $B \cap \mathcal{Q}$ ctble. $\square$

Side comment.

(1) $\exists$ fcns on $[0,1]$ which are discont at every x.

e.g. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

(2) what happens if f is increasing $x < y \Rightarrow f(x) \leq f(y)$?

In this case f has $\leq$ ctble many discont.
Pf similar to above. $\square$

Moreover, the relationship between measures on $[0,1]$
and ~~the~~ dist. fcns tells us this.

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non σ-finite space

$([0,1], \mathfrak{m} = \mathcal{P}[0,1],$ counting msre $\mu)$

$\mu(A) = |A|$

Folland 1.18. Variant for Leb. msre.

(a) $\forall E \subseteq \mathbb{R}$, ($E$ need not be msbl)

$\forall \varepsilon > 0$, $\exists$ open set $\mathcal{O}$ s.t. $E \subseteq \mathcal{O}$,

$\underline{m}(\mathcal{O}) \leq m^*(E) + \varepsilon$

Leb. msre

Pf. case (1) $m^*(E) = \infty$, then $\mathcal{O} = \mathbb{R}$

case (2) $m^*(E) < \infty$. By defn, $\exists$ open intervals $I_1, I_2, \dots$

$E \subseteq \bigcup I_i$ s.t. $\sum_{i=1}^{\infty} |I_i| \leq m^*(E) + \varepsilon$.

Let $\mathcal{O} = \bigcup_{i=1}^{\infty} I_i$. $\mathcal{O}$ open. $E \subseteq \mathcal{O}$. and

$m(\mathcal{O}) \underset{\text{sub.}}{\leq} \sum_{i=1}^{\infty} |I_i| \leq m^*(E) + \varepsilon$ $\square$

b'. Let $m^*(E) < \infty$ (E need not be msbl)

Then $\exists$ a countable intersection of open

sets, $\underbrace{\bigcap_{i=1}^{\infty} \mathcal{O}_i}_{G_\delta \text{ sets}}$ s.t. $E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$ and

$m(\bigcap_{i=1}^{\infty} \mathcal{O}_i) = m^*(E)$.

pf. by (a) choose $\mathcal{O}_k$ open $E \subseteq \mathcal{O}_k$, +

$m(\mathcal{O}_k) \leq m^*(E) + 1/k$. Consider $\bigcap_{i=1}^{\infty} \mathcal{O}_i$.

$E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$. Finally $\forall k$,

$\underline{m(\bigcap_{i=1}^{\infty} \mathcal{O}_i)} \leq m(\mathcal{O}_k) \leq \underline{m^*(E)} + 1/k$

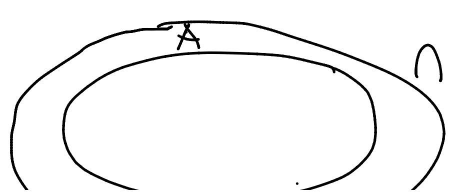
Since true $\forall k$, $m(\bigcap \mathcal{O}_i) \leq m^*(E)$

---

b''. Thm. E is m^*-msble iff

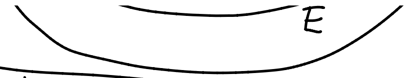
$\exists$ ctble intersection of open sets $\bigcap_{i=1}^{\infty} \mathcal{O}_i$ s.t. $E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$

$m^*(\underbrace{\bigcap_{i=1}^{\infty} \mathcal{O}_i \setminus E}_{A}) = 0$



not same as $m(\bigcap \mathcal{O}_i) = m^*(E)$

$m(\bigcap O_i) = m(E)$



$\Leftarrow$ $m^*(A) = 0 \Rightarrow A$ is m^*-msble

$\Rightarrow \bigcap_{i=1}^{\infty} O_i \setminus A$ m^*-mble $\Rightarrow E$ is m^* mble

$\Rightarrow$ By 5', $\exists$ open sets O_i s.t. $E \subseteq \bigcap O_i$

$m^*(\bigcap O_i) = m^*(E)$. On other hand

since E is mble,

$$m^*\left(\bigcap_{i=1}^{\infty} O_i\right) = m^*\left(\bigcap O_i \cap E\right) + m^*\left(\underbrace{\bigcap O_i \cap E^c}_{A}\right)$$

$$= m^*(E) + m^*(A).$$

$\Rightarrow m^*(A) = 0.$