7 Folland 1.8

$\varliminf_{n\to\infty} a_n$, $\quad \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \underbrace{\sup_{k\ge n} a_k}_{\text{dec in } n}$

$1, -1, 1, -1, 1, -1 \ldots$

$$\varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \underbrace{\inf_{k\ge n} a_k}_{\text{inc. in } n}$$

$$\varliminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$$

$= \{x : x \text{ is in all of the } E_n\text{'s except finitely many}\}$

$$\mu(\varliminf E_n) \le \varliminf_{n\to\infty} \mu(E_n)$$

~~basic~~ baby version of Fatou's Lemma

Pf $\mu\left(\bigcap_{k=n}^{\infty} E_k\right) \le \mu(E_j) \quad \forall j \ge n.$

$\Rightarrow \underbrace{\mu\left(\bigcap_{k=n}^{\infty} E_k\right)}_{\text{inc in } n} \le \inf_{j\ge n} \mu(E_j) \le \varliminf \mu(E_j)$

$$\mu(\varliminf E_k) = \mu\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) \underset{\text{cont. from below}}{=} \lim_{n\to\infty} \mu\left(\bigcap_{k=n}^{\infty} E_k\right) \le \varliminf_{n\to\infty} \mu(E_n)$$

$\square$

$\mu(\overline{\lim_{n\to\infty}} E_n) \ge \overline{\lim_{n\to\infty}}\, \mu(E_n)$ if

$\mu(\bigcup_{i=1}^{\infty} E_i) < \infty$. $\longrightarrow$?

Ex ([0, ∞), Leb. measure).

$E_n = [n, n+1]$ LHS 0 RHS = 1

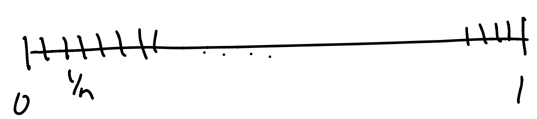
$\emptyset$

Hint: let $F_k = \bigcup_{j=1}^{\infty} E_j \setminus E_k$

$\mu(\underline{\lim}\, F_k) \le \underline{\lim}\, \mu(F_k)$ by part 1.

Riemann intg.

$\int_0^1 f\,dx$

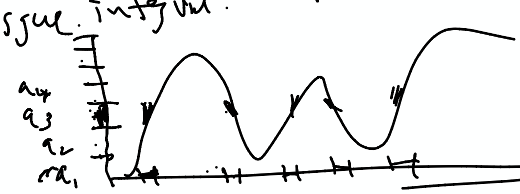


$\sum_{i=1}^{n} f(\frac{i}{n}) \frac{1}{n}$ let $n \to \infty$.

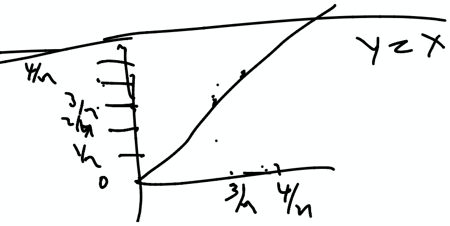
Lebesgue integral. Break the y-axis into small piece

$f(x) = I_Q(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$

not Riemann intg



$\sum_{i=1}^{n} m\{x : f(x) \in [a_2, a_3]\}\, a_3$



Unfortunately I will need to cancel today's class ~~Lecture~~ which should start at 10:45. I apologize for this. On thursday, we do some exercises.

A. pg 7 of Lecture notes:

Example of an alg. not a σ-alg.

$X = \mathbb{N}$, $\mathcal{Q} = \{ A \subseteq \mathbb{N} : |A| < \infty$ or $\underbrace{|\mathbb{N} \setminus A| < \infty}_{A \text{ cofinite}} \}$

$\mathcal{Q}$ is a ~~σ~~ alg. not a σ-alg.

$E =$ even integers. $E \notin \mathcal{Q}$ $\quad E = \bigcup_{i=1}^{\infty} \underbrace{\{2i\}}_{\in \mathcal{Q}} \Rightarrow \mathcal{Q}$ not a σ-alg.

Pg 8 L.N. $\quad \mathcal{F}_1 = \{X, \phi\}$, $\mathcal{F}_2 = \{$all ctable ~~and~~ or cocountable sets$\}$

X set

$\mathcal{F}_3 = \mathcal{P}(X) = \{$all subsets of $X\}$.

$\mathcal{F}_1, \mathcal{F}_3$ trivially σ-alg. $\quad \mathcal{F}_2$ is a σ-alg.

Pf $\ni X, \phi$ ✓ $\quad A \in \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_2$. (symmetry)

$A_1, A_2, A_3 \dots \in \mathcal{F}_2$. NTS $\bigcup A_i \in \mathcal{F}_2$

case 1 all A_i's are ctble. $\Rightarrow \bigcup A_i$ ctble $\Rightarrow \bigcup A_i \in \mathcal{F}_2$.

case 2 at least one A_i is cocountable. A_n.

Then $\bigcup A_i$ cocountable since

$$|X \setminus \bigcup A_i| \le |X \setminus A_n| \qquad \square$$

$\mathcal{F}_1 = \mathcal{F}_2 \iff |X| = 1$.

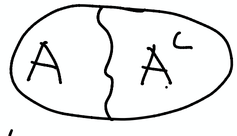
$\Leftarrow$ $|X| = 1$, only 1 σ-alg.

$\Rightarrow$ $|X| \ge 2$. choose $x \in X$. $\{x\} \in \mathcal{F}_2$ $\{x\} \notin \mathcal{F}_1$

$\mathcal{F}_2 = \mathcal{F}_3 \iff X$ ctble.

$\Leftarrow$ Easy $\quad \Rightarrow$. assume X not ctble.

WTS $\mathcal{F}_2 \neq \mathcal{F}_3$.



X want to find $A \subseteq X$

with A unctble, A^c unctble. Such an $A \in \mathcal{F}_3$; $A \notin \mathcal{F}_2$

$X \times \{0\} = \{ (x,0) : x \in X \}$. $\quad X \times \{1\} = \{ (x,1) : x \in X \}$.

$|X \times \{0\}| = |X| = |X \times 1|$. $\quad (X \times \{0\}) \cup (X \times \{1\}) \longrightarrow X$

$\exists$ $\underline{f \text{ bij}}$ since $|X| = \infty$.

$A = f(X \times \{0\})$.

Folland 1.3.

$\mathcal{M}$ is an inf. σ-alg. (ie. $\exists\ \infty$ # of sets in $\mathcal{M}$).

(a) $\mathcal{M}$ contains a sequence of nonempty disjoint sets. $\underline{B_1, B_2, B_3 \ldots}$ disjoint nonempty

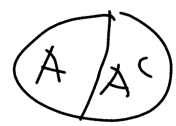
(b) $\text{card}(\mathcal{M}) \geq c$ ($=$ card. of real #'s)

(a) call $A \in \mathcal{M}$ finite if

$$|\{E \in \mathcal{M} : E \subseteq A\}| < \infty \qquad \text{inf o.w.}$$

Note X is inf. by assumption. Choose $A \in \mathcal{M}$, $A \neq \emptyset$, $A \neq X$.

crucial observ. is either A or A^c is infinite

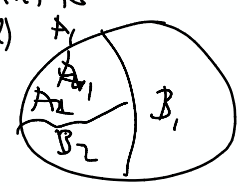


$\big(B \in \mathcal{M} \Rightarrow B = \underset{\in \mathcal{M}}{(B \cap A)} \cup \underset{\in \mathcal{M}}{(B \cap A^c)} \big)$

Let A_1 piece which is infinite, B_1 inf. or finite

A_1 inf. Break A_1 into A_2, B_2 partition (nontrivial)

such that A_2 is inf.



Keep going $\Rightarrow B_1, B_2, B_3 \ldots$ disjoint + nonempty.

(b) $c = |[0,1]| = |\{0,1\}^{\mathbb{N}}|$

$\{0,1\}^{\mathbb{N}}$ = infinite binary sequences

Find an $\underline{\text{injective}}$ map from $\{0,1\}^{\mathbb{N}}$ to $\underline{\mathcal{M}}$

$x \neq y \Rightarrow f(x) \neq f(y)$ $\qquad \Rightarrow |\mathcal{M}| \geq |\{0,1\}^{\mathbb{N}}|$. done

$0110100100010000 \longrightarrow B_2 \cup B_3 \cup B_5 \cup B_8 \cup B_{12}$

1 2 3 4 5 6 7 8 9 10 11 12

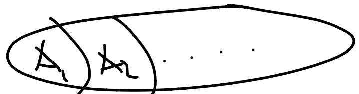
f inj since B_is are disjoint.

·PS 14 Lecture notes.

(X, m, μ) is σ-finite, then $\mathcal{C}$ (= atoms) is at most ctble.

Pf. σ-finite $X = \bigcup_{i=1}^{\infty} A_i$ s.t. $\mu(A_i) < \infty$

If we can show $\forall i$ $|\mathcal{C} \cap A_i|$ ctble, then done.



NTS $\mu(B) < \infty$, then $B \cap \mathcal{C}$ ctble

Pf let $F_k = \{x \in B \cap \mathcal{C} : \mu\{x\} \geq \frac{1}{k}\}$.

claim: $|F_k| < \infty$. Since if infinite,

$\mu(B) \geq \mu(F_k) \geq \frac{1}{k}\infty = \infty$. ↯

~~Also~~ $B \cap \mathcal{C} = \bigcup_{k=1}^{\infty} F_k \Rightarrow B \cap \mathcal{C}$ ctble. $\square$

Side comment

(1) $\exists$ fcns on $[0,1]$ which are discont at every x.

eg. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

(2) what happens if f is increasing $x<y \Rightarrow f(x) \leq f(y)$?

In this case f has $\leq$ ctble many discont.

Pf similar to above. $\square$

Moreover, the relationship between measures on $[0,1]$ and ~~the~~ dist. fcns tells us this.

---

non σ-finite space

$([0,1],\ m = \mathcal{P}[0,1],\ \text{counting measure } \mu)$

$\mu(A) = |A|$

Folland 1.18. Variant for Leb. msre.

(a) $\forall E \subseteq \mathbb{R}$, (E need not be msbel

$\forall \varepsilon > 0$, $\exists$ open set $\mathcal{O}$ s.t $E \subseteq \mathcal{O}$,

$\underline{m}(\mathcal{O}) \le m^*(E) + \varepsilon$

Leb. msre

pf. case (1) $m^*(E) = \infty$, then $\mathcal{O} = \mathbb{R}$

Case (2) $m^*(E) < \infty$. By defn, $\exists$ open intervals I_1, I_2 - -

$E \subseteq \cup I_i$ s.t $\sum_{i=1}^{\infty} |I_i| \le m^*(E) + \varepsilon$.

let $\mathcal{O} = \bigcup_{i=1}^{\infty} I_i$. $\mathcal{O}$ open. $E \subseteq \mathcal{O}$. and

$m(\mathcal{O}) \underset{\text{sub.}}{\le} \sum_{i=1}^{\infty} |I_i| \le m^*(E) + \varepsilon$ $\square$

b'. Let $m^*(E) < \infty$ (E need not be msbel

Then $\exists$ a countable intersection of open

sets $\underbrace{\bigcap_{i=1}^{\infty} \mathcal{O}_i}_{G_\delta \text{ sets}}$ s.t $E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$ and

$m(\bigcap_{i=1}^{\infty} \mathcal{O}_i) = m^*(E)$.

pf. by (a) choose $\mathcal{O}_k$ open $E \subseteq \mathcal{O}_k$, +

$m(\mathcal{O}_k) \le m^*(E) + 1/k$. Consider $\bigcap_{i=1}^{\infty} \mathcal{O}_i$.

$E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$. Finally $\forall k$,

$\underline{m(\bigcap_{i=1}^{\infty} \mathcal{O}_i)} \le m(\mathcal{O}_k) \le \underline{m^*(E)} + 1/k$

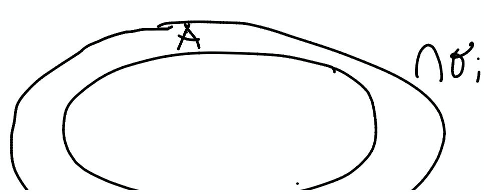
Since true $\forall k$, $m(\cap \mathcal{O}_i) \le m^*(E)$

---

b''. Thm. E is m^*-msble iff

$\exists$ ctble intersection of open sets $\bigcap_{i=1}^{\infty} \mathcal{O}_i$ s.t $E \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_i$

$m^*(\underbrace{\bigcap_{i=1}^{\infty} \mathcal{O}_i \setminus E}_{A}) = 0$



not same as $(\cap \mathcal{O}_i) = m^*(E)$

$m(\bigcap O_i) = m(E)$

$\Leftarrow$ $m^*(A) = 0 \Rightarrow A$ is m^*-measurable

$\Rightarrow \bigcap_{i=1}^{\infty} O_i \setminus A$ m^*-mble $\Rightarrow E$ is m^* mble

$\Rightarrow$ By 5', $\exists$ open sets O_i s.t. $E \subseteq \bigcap O_i$

$m^*(\bigcap O_i) = m^*(E)$. On other hand

Since E is mble,

$$m^*\Big(\bigcap_{i=1}^{\infty} O_i\Big) = m^*\Big(\bigcap O_i \cap E\Big) + m^*\Big(\underbrace{\bigcap O_i \cap E^c}_{=A}\Big)$$

$$= m^*(E) + m^*(A).$$

$\Longrightarrow m^*(A) = 0.$

1. Folland 1.130

Let E be Leb. msble and $0 < m(E) < \infty$.

$\forall \alpha < 1$ $\exists I$ interval s.t $m(E \cap I) \geq \alpha m(I)$

"I is mostly filled up by E"

[$m(E) = \infty$, replace E by $E \cap [-n,n]$]

Let $\alpha < 1$. Choose intervals $J_1, J_2, \ldots$ cover E ($E \subseteq \bigcup J_i$), $\sum_{i=1}^{\infty} |J_i| \leq$ ~~$\frac{m(E)+\varepsilon}{}$~~ $\frac{m(E)}{\alpha}$ $\quad m(E) > 0$

$$\sum_{i=1}^{\infty} m(E \cap J_i) \underset{\text{subad}}{\geq} m(E) \geq \sum_{i=1}^{\infty} \alpha m(J_i)$$

$\Rightarrow \exists i_0$ s.t $m(E \cap J_{i_0}) \geq \alpha m(J_{i_0})$ $\square$

---

(1) Int. (2) allows appreciation of 1.31.

(3) crucial for slide presentation

If $I \subseteq A$, $I = [a,b]$, $a < b$. $\Rightarrow m(A) > 0$.

Converse: If $m(A) > 0$, does A contain an interval?

No. $A = [0,1] \setminus \mathbb{Q}$. $m(A) = 1$. But A contains no interval since contains no rationals

Converse if A closed.

Still no. – Fat Cantor sets.

(a) Variant of the Cantor set but remove middle $1/4$'s rather than $1/3$'s. Ex. ✓

(b) $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, q_4, \ldots\}$

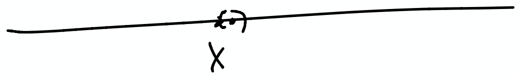
$$A = [0,1] \setminus \bigcup_{i=1}^{\infty} \left(q_i - \frac{1}{2^{i+1} 10}, q_i + \frac{1}{2^{i+1} 10}\right)$$

A closed ✓ A contains no interval (since contains no rat.)

$m(A) > 0$? $m\left(\bigcup_{i=1}^{\infty} J_i\right) \leq \sum_i |J_i| = \sum_{i=1}^{\infty} \frac{1}{2^i 10} = \frac{1}{10}$.

$\Rightarrow m(A) \geq 9/10$.

$x \in A$



1.31. $m(E) > 0$. Then $E - E = \{e_1 - e_2 : e_1, e_2 \in E\}$ contains an interval.

$\boxed{\text{bigger than } E}$

$\supseteq E - e_1$

Sol. By 1.30. $\exists$ interval I s.t. $m(E \cap I) \geq \frac{3}{4} \cancel{m(E)}\, m(I)$

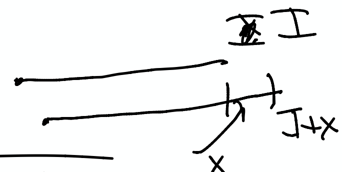
$A := E \cap I \quad B = (E \cap I) + x$

If $A \cap B \neq \emptyset \Rightarrow e_1 = e_2 + x$ some $e_1, e_2 \in E \Rightarrow x \in E - E$.

If $A \cap B = \emptyset$, then $m(A \cup B) = 2\, m(E \cap I) \geq \frac{3}{2} m(I)$.

also $m(A \cup B) \leq m(I \cup (I + x))$

$\leq \frac{3}{2} m(I)$ If $|x| \leq \frac{|I|}{2}$.



Hence if $|x| \leq \frac{|I|}{2}$, then $A \cap B \neq \emptyset \Rightarrow x \in E - E$.

$[-|I|/2, \frac{|I|}{2}] \subseteq E - E$. $\square$

1.33. $\exists$ Borel set $A \subseteq [0,1]$ s.t.

$\forall$ intervals I $\quad 0 < m(A \cap I) < m(I)$

A is everywhere

$[0,1] \setminus A$ is everywhere

Recall ~~μ, ν~~ if μ, ν finite msres on $\mathbb{X}$ s.t $\mu(\mathbb{X}) = \nu(\mathbb{X})$ and

$\mu = \nu$ on a π-system $\mathcal{D}$ $(\mu(I) = \nu(I)\ \forall\ I \in \mathcal{D})$

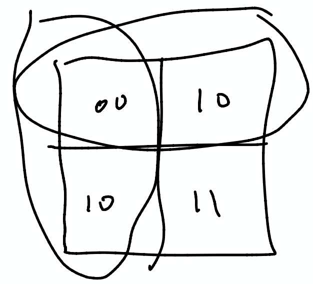
then $\mu = \nu$ on $\sigma(\mathcal{D})$.

---

need a π-system : maybe $\mu = \nu$ on $\mathcal{E}$.
then $\mu = \nu$ on $\sigma(\mathcal{E})$. "Lin trans. determined by its values on a basis".

$\mathbb{X} = \{ (0,0), (1,0), (0,1), (1,1) \}$

μ	$1/4$	$1/4$	$1/4$	$1/4$
ν	$1/2$	0	0	$1/2$



$\mathcal{E} = \{ \{(0,0)\ (1,0)\},\ \{(0,0), (0,1)\} \}$

$\mu = \nu$ on $\mathcal{E}$. $\sigma(\mathcal{E}) = \mathcal{P}(\mathbb{X})$.

What is happening.

$\mathbb{X}, \mathbb{Y}$ be indep rvs. $P(\mathbb{X}=1) = P(\mathbb{X}=0) = P(\mathbb{Y}=1) = P(\mathbb{Y}=0)$
$= 1/2$.

$(\tilde{\mathbb{X}}, \tilde{\mathbb{Y}})$. $P(\tilde{\mathbb{X}}=1) = P(\tilde{\mathbb{X}}=0) = 1/2$. $\tilde{\mathbb{Y}} = \tilde{\mathbb{X}}$. $\Rightarrow P(\tilde{\mathbb{Y}} = 1) = P(\tilde{\mathbb{Y}}=0) = 1/2$

dist of $(\mathbb{X}, \mathbb{Y})$

00	$1/4$
10	$\vdots$
01	$\vdots$
11	$\vdots$

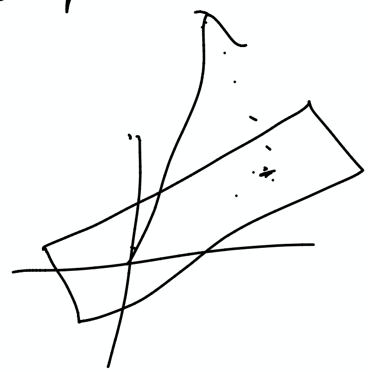
dist of $(\tilde{\mathbb{X}}, \tilde{\mathbb{Y}})$

00	$1/2$
10	0
01	0
11	$1/2$

$\{(0,0)\ (1,0)\}$
$= \{\mathbb{Y} = 0\}$
$\{\tilde{\mathbb{Y}} = 0\}$ $\square$

$L^p(X, \mathcal{M}, \mu) = \{ f : X \to \mathbb{R} : \int |f|^p < \infty \}$.

$\|f\| = \left(\int |f|^p \right)^{1/p}$. Banach space.



$p=2$ special Hilbert space

Look like $\mathbb{R}^n$ analytic geom

Same picture for L^2 except inf. dim.

$V \cdot W$ dot prod

$f \cdot g := \int f(x) g(x) \, d\mu(x)$. $\mathbb{R}^n$ special case.

$\mathbb{R}^3$ $X = \{1,2,3\}$. μ each pt is 1.

$L^2(X, \mu) = \mathbb{R}^3$ every way.