L. notes pg 41 → very general

$f: [0,1] \to \mathbb{R}$. The set of pts where f is cont is a Borel set.

Pf. Show stronger. It is in fact a G_δ set; i.e. it is a ctble intersection of open sets.

C = pts. of cont of f.

$$C = \bigcap_{n=1}^{\infty} \Bigg(\underbrace{\bigcup_{\substack{I \text{ open interval} \\ \operatorname{diam}(f(I)) \le 1/n}} I}_{\text{open}} \Bigg)$$

$f(I) = \{f(x) : x \in I\}$

$\operatorname{diam} A = \sup\{|x-y| : x, y \in A\}$

Pf $\subseteq$

let $x \in C$. Fix $n \ge 1$. NTS $x \in \bigcup$.

Choose $\delta > 0$ s.t. $\forall y \in (x-\delta, x+\delta)$

$|f(y) - f(x)| \le \frac{1}{2n}$.

i.e. $f(\underbrace{(x-\delta, x+\delta)}_{I}) \subseteq \underbrace{[f(x) - \frac{1}{2n}, f(x) + \frac{1}{2n}]}_{\operatorname{diam} \frac{1}{n}}$. $\square$

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(1) Does there exist $f: [0,1] \to \mathbb{R}$ s.t. f is cont precisely at the irrationals?

(2) . . . . rationals?

(1) we did. Yes. Take a measure μ on $[0,1]$ concentrated on rationals. $\mathbb{Q} = (q_1, q_2, q_3, \ldots)$

$\mu(\{q_i\}) = \frac{1}{2^i}$. $\mu[0,1] = 1$. Dist fcn of μ works.

(2) NO. The previous exercise $\Longrightarrow$ if $\exists$ a fcn cont. only on $\mathbb{Q}$, then $\mathbb{Q}$ must be a ctble intersection of open sets.

$$[0,1] \setminus \mathbb{Q} = \bigcap_{i=1}^{\infty} ([0,1] \setminus \{q_i\})$$

$\mathbb{Q}$ is not a ctble $\cap$ of open sets. Baire Category Thm. $\square$

L.N. pg 56.

1. ~~f_n~~ $0 \le f_n \le 1$, $f_n \to 0$ uniformly,

but $\int f_n \not\to 0$.

$[0,\infty)$ Leb msre. $f_n = \frac{1}{n} I_{[0,n]}$

1st 2 cond. obv. $\int f_n(x)\,dx = \frac{1}{n} \cdot n = 1$.

Since $\int f_n \not\to 0$, The LDC Thm says there cannot exist a fcn $g \in L^1([0,\infty))$ which dominates all fn's ($|f_n| \le g \ \forall n$). Let's verify "with hands" there is no such fcn. assume there is. Say g is. $|f_n| \le g \Rightarrow g(x) \ge \frac{1}{n}$ on $[n-1,n]$.

$$\int g = \sum_{m=1}^{\infty} \int_{m-1}^{m} g(x)\,dx \ge \sum_{m=1}^{\infty} \frac{1}{m} = \infty$$

Example cannot occur on a finite measure space $(X, \mathcal{B}, \mu)$ $\mu(X) < \infty$. $|f_n| \le 1$ $f_n \to 0$ uniformly.

then $\int f_n \,d\mu \to 0$. Why: LDC with $g(x) = 1 \ \forall x$.

Constants are integrable on finite spaces but not on inf. spaces

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Find an example, ~~$|f_n|$~~ $0 \le f_n \le 1$

$f_n \to 0$ unif., $\int f_n \to 0$. but $\nexists$ a dom. fcn g.

I.e. $\nexists g$: $|f_n| \le g \ \forall n$ and $g \in L^1(\ )$

$[0,\infty)$, Leb msre

$f_n = \frac{1}{n} I_{[n,n+1]}$. $\int f_n = \frac{1}{n} \to 0$. But no g.

why: $g \ge |f_n| \Rightarrow g(x) \ge \frac{1}{n}$ on $[n,n+1]$ $\forall n$.

$$\int_0^\infty g(x) = \sum_{n=0}^{\infty} \int_n^{n+1} g(x)\,dx \ge \sum_{n=0}^{\infty} \int_n^{n+1} 1/n \,dx$$

$$= \sum_0^{\infty} \frac{1}{n} = \infty. \quad \square$$

$([0,1], \cancel{m})$. Find $f_n \geq 0$.

$f_n(x) \to 0 \quad \forall x$. as $n \to \infty$.

$\int f_n \not\to 0$ but $\nexists$ dom. g.

$f_n = \cancel{n I_{[\frac{1}{n}}} \quad n \; I_{[\frac{1}{n+1}, \frac{1}{n}]}$.

Folland. Ex 2.3. f_n seq. of fns

$\{x : f_n(x) \text{ converges}\}$ msble.

Did before? No. $(f_n) + f$ $\quad \{x : f_n(x) \to f(x)\}$ msble

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meaning of converging.

(1) ∞ is allowed

$= \{x : \overline{\lim} f_n(x) = \underline{\lim} f_n(x)\}$. Generally,

f, g msble $\quad \{x : f(x) = g(x)\}$ is msble

why? $\{x : f(x) = g(x)\} = \{x : f(x) - g(x) = 0\}$

$= (f(x) - g(x))^{-1}\{0\}$ msble.

(2) converges means to a finite #.

$\{x : f_n(x) \text{ converges}\} = \{x : f_n(x) \text{ Cauchy seq.}\}$

$= \{x : \bigcap_{n \geq 1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k, l \geq m}^{\infty} \{x : |f_k(x) - f_l(x)| < 1/n\}$..

Folland 2.4.

assume $\{x : f(x) \in (r, \infty)\}$ msble $\forall r \in \mathbb{Q}$.

$\qquad \rotatebox{90}{$=$}$ $\qquad$ WTS f msble.

$f^{-1}((r,\infty))$

Sol. $\forall x \in \mathbb{R}, \quad f^{-1}((x, \infty)) = \bigcup_{\substack{r > x \\ r \in \mathbb{Q}}} \underbrace{f^{-1}((r, \infty))}_{\in \mathcal{M}}$ by assump. ctble union

$\supseteq$

$\subseteq$ Let $w \in$ LHS $\Rightarrow f(w) > x$. choose $r \in \mathbb{Q}$ $\quad x < r < f(w)$

$f(w) > r \Rightarrow w \in f^{-1}((r,\infty))$. $\square$

2.30. $\int_0^k x^n \left(1-\frac{x}{k}\right)^k dx \longrightarrow n!$ as $k \to \infty$ $\forall n$.

Sol. LHS $= \int_0^\infty \underbrace{x^n \left(1-\frac{x}{k}\right)^k I_{[0,k]}(x)}_{\geq 0} dx$.

What does integrand $\longrightarrow$ as $k \to \infty$ for fixed x.

$x^n \lim_{k\to\infty} \left(1-\frac{x}{k}\right)^k = x^n e^{-x}$. If we could apply LDC,

then $\longrightarrow \int_0^\infty x^n e^{-x} dx = n!$ by induction n

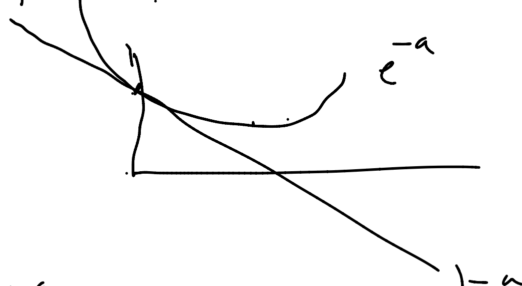
we do have a dom fcn. $g(x) = x^n e^{-x} \in L^1$

$|f_k(x)| \leq g(x)$ $\forall x$ $\forall n$. Ie. $x^n \left(1-\frac{x}{k}\right)^k \leq x^n e^{-x}$ for $x \in [0,k]$

ie. $\left(1-\frac{x}{k}\right)^k \leq e^{-x}$ for $x \in [0,k]$

Key fact. $a \in [0,1]$ $(1-a) \leq e^{-a}$ Given key fact

$\Big(1-\underbrace{\frac{x}{k}}_{a}\Big)^k \leq \left(e^{-\frac{x}{k}}\right)^k = e^{-x}$



$1-a$, e^{-a} agree at 0.

NTS. $(e^{-a})' \geq (1-a)'$ ie.

$-e^{-a} \geq -1$ ie. $e^{-a} \leq 1$ $a \geq 0$