F. 2.14

$f \in L^+(X, \mathcal{B}, \mu)$, define $\lambda(E) = \int_E f\,d\mu$,

then λ is a measure. $\lambda(\emptyset) = \int_\emptyset f\,d\mu = 0$.

Let $E_1, E_2, \dots$ disjoint msre sets.

$\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \int_{\cup E_i} f\,d\mu = \int_X f I_{(\cup E_i)}$

disj. $= \int f \sum_{i=1}^{\infty} I_{E_i} = \int \sum_{i=1}^{\infty} f I_{E_i}$

$\overset{MCT}{=} \int \lim_{n\to\infty} \sum_{i=1}^{n} f I_{E_i} \overset{MCT}{=} \lim_{n\to\infty} \int \sum_{i=1}^{n} f I_{E_i}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \int f I_{E_i} = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda(E_i) = \sum_{i=1}^{\infty} \lambda(E_i)$. □

Note given $f \ge 0$, μ, we obtained λ

$(f, \mu) \longrightarrow \lambda$

Note $\mu(A) = 0 \Rightarrow \lambda(A) = 0$.

triv. $\lambda(A) = \int_A f = \int \underbrace{f I_A}_{=0 \text{ a.e.}} = 0$.

?? If you give me λ, μ s.t $\mu(A) = 0 \Rightarrow \lambda(A) = 0$, must λ arise as above. Ie $\exists f$ s.t. $(f, \mu) \longrightarrow \lambda$?

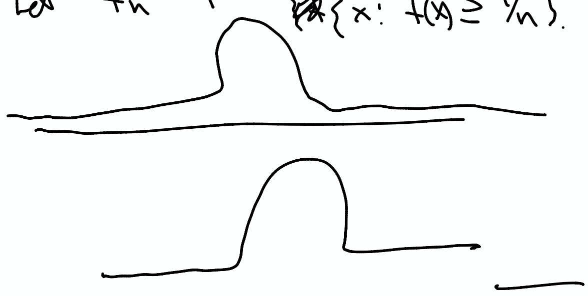
Yes Radon Nykdym Thm

F 2.16 $f \ge 0$. $\int f\,d\mu < \infty$.

Then $\forall \varepsilon > 0$, $\exists E$ s.t $\mu(E) < \infty$, but $\int_E f\,d\mu \ge \int f\,d\mu - \varepsilon$.

Sol (MCT).

Let $f_n = f I_{\{x:\, f(x) \ge 1/n\}}$.



$\int_{\{x:\, f(x) \ge \frac{1}{n}\}} f = \int f_n$. Note $f_n(x) \nearrow f(x)\ \forall x$

$\underset{MCT}{\Rightarrow} \int f_n \nearrow \int f < \infty$

$\Rightarrow \exists N$ s.t $\int f_N \ge \int f - \varepsilon$.

$\int f \ge \int f - \varepsilon$.

$\{x:\, f(x) \ge 1/n\} \rightarrow$ need finite msre

must be ~~since~~ O.W.

$\infty > \int f \ge \frac{1}{n} \mu\{x:\, f(x) \ge \frac{1}{n}\}$

$\ge \int \frac{1}{n} I_{\{f(x) \ge \frac{1}{n}\}}$ $< \infty$.

F. 22. $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{c.m.})$

Fatou. $f_j(x)$ $a_j(i) = a_{ij}$

i^{th} term of j^{th} seq. Fatou

$\{a_{ij}\}_{i,j}$ $a_{ij} \geq 0$

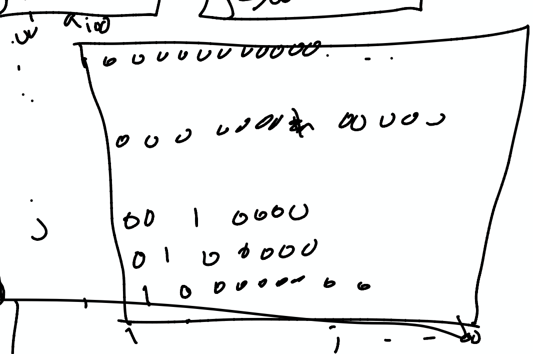
$$\boxed{\sum_i \underline{\lim}_{j\to\infty} a_{ij}} \leq \boxed{\underline{\lim}_{j\to\infty} \sum_i a_{ij}}$$

$$\int \underline{\lim}_{n\to\infty} f_n(x) \leq \underline{\lim}_{n\to\infty} \int f_n(x)$$

$$a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In particular if $\forall i$ $\lim_{j\to\infty} a_{ij} = a_{i\infty}$

then $0 \leq \boxed{\sum a_{i\infty}} \leq \boxed{\underline{\lim}_{j\to\infty} \sum_i a_{ij}}$



~~Fubini~~ "failure of Fubini"

Why all assumptions in Fub. needed?

1. Seen σ-finiteness needed.

$([0,1]$ Leb$) \times ([0,1], \text{c.m})$

$D = \text{diag}$

F. 47. Note $\mathbb{N}$ is an $\underline{\underline{\text{inf}}}$ set s.t. $\forall y \in \mathbb{N}$, $\{x : x < y\}$ finite set.

modification

Let X be an unctble set, which is 'Linearly ordered' s.t. $\forall y \in X$,

$\{x \in X : x < y\}$ is a ctble set.

X = $\underline{\text{first}}$ $\underline{\text{unctble}}$ ~~or Ai~~ ~~words~~ ordinal

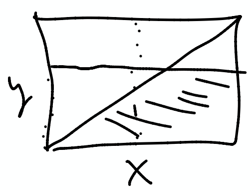
Let $(X, \mathcal{M}, \mu)$ X ~~Fat~~ ~~tot~~ unc. FUO.

$\mathcal{M}$ = all ctble or cocountable sets

ctble set

$$\mu(A) = \begin{cases} 0 & A \text{ ctbe} \\ 1 & A \text{ coctble} \end{cases}$$

✓ msre.. finite $E \subseteq X \times X$ $E = \{(x,y) : y < x\}$.



2 ~~intg~~ iterated $\int$s

$$\int_X \mu(E_x)\, d\mu(x) = \int_X \mu(\{y : y < x\})\, d\mu(x) = \int_X 0 = 0$$

$$\int_Y \mu(E_y)\, d\mu(y) = \int_Y \mu(\underbrace{x : y < x}_{})\, d\mu(y) = \int 1\, d\mu = 1.$$

cocountable set. since

$\{x : y < x\}^c = \{x : x \le y\}$ = ctble

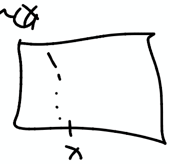
Problem. E is not msble. !!

$$\underline{E_x} = \{y : (x,y) \in E\} \overset{\text{ourcase}}{=} \underline{\{y : y < x\}}$$

$$\int \mu(E_x)\, d\mu(x) = \int \boxed{\mu(y : y < x)}\, d\mu(x)$$

ctble set

0



F.48 $X = Y = \mathbb{N}$, $\mathcal{M}$ all subsets

$\mu = \nu =$ c.m.

$$f(m,n) = \begin{cases} 1 & m = n \\ -1 & m = n+1 \\ 0 & \text{o.w.} \end{cases}$$

compute both it. integrals

$$\sum_m \underbrace{\sum_n f(m,n)} = \ ?$$

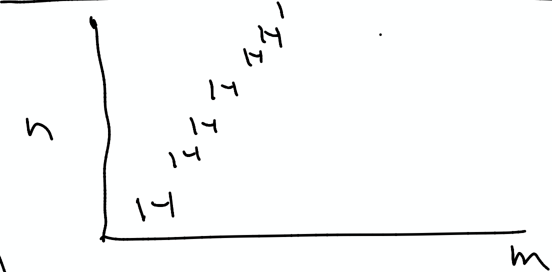
$$\sum_n f(m,n) = \begin{cases} 0 & m \neq 1 \\ 1 & m = 1 \end{cases}$$

$$\to \sum_m \to 1$$

$$\sum_n \sum_m f(m,n). \ ?$$

$$\sum_{m=1}^{\infty} f(m,n) = \{ 0 \ \forall n \}$$

$$= \to 0.$$



Problem: $f \notin L^1$ $\quad \sum_{n,m} |f(n,m)| = \infty$

F. 38. $f_n \to f$ in msre, $g_n \to g$ in msre,

then (1) $\underline{f_n + g_n \to f+g}$ in msre (2) $f_n g_n \to fg$ in msre if finite measure space.

(1) Fix $\varepsilon > 0$. $\mu(|(f_n+g_n)-(f+g)| \geq \varepsilon) \xrightarrow{?} 0$.

$\leq \mu\{x : |f_n - f| + |g_n - g| \geq \varepsilon\} \leq$

$\Delta \leq$ $\cap$

$\mu(\{x : |f_n - f| \geq \varepsilon/2\} \cup \{x : |g_n - g| \geq \varepsilon/2\})$

$\leq \underbrace{\mu(x : |f_n - f| \geq \varepsilon/2)}_{\to 0 \text{ as } n \to \infty} + \underbrace{\mu(x : |g_n - g| \geq \varepsilon/2)}_{\to 0 \text{ as } n \to \infty}$

by assumption.

Fix $\varepsilon > 0$. $\mu\{x : |f_n g_n - fg| \geq \varepsilon\} \overset{?}{\to} 0$

$= \mu\{x : |f_n g_n - f g_n + f g_n - fg| \geq \varepsilon\}$

$\underset{\Delta}{\leq} \mu\{x : |f_n g_n - f g_n| + |f g_n - fg| \geq \varepsilon\}$

$\leq \mu(x : |f_n g_n - f g_n| \geq \varepsilon/2)$ + $\mu(x : \underbrace{|f g_n - fg|}_{|f|\,|g_n - g|} \geq \varepsilon/2)$

$|g_n|\,|f_n - f|$

(same as before 2 steps)

2nd term goes to 0

Fix $\delta > 0$. Choose M s.t. $\mu(:|f(x)| \geq M\} < \delta$. $\overset{?}{\text{why can do}}$

$\bigcap_M \{x : |f(x)| \geq M\} = \phi$

$0 = \mu(\phi) = \mu(\bigcap_M \quad) = \lim_{M \to \infty} \mu(x : f(x) \geq M)$ (finite m.s.)

ε, δ, M.

$\mu(|f|\,|g_n - g| \geq \varepsilon/2) \leq \mu\{\{|f| \geq M\} \cup \{|g_n - g| \geq \frac{\varepsilon}{2M}\}\}$

$\boxed{\text{If } x \notin \cup, \text{ then } |f(x)| < M,\ |g_n - g| < \varepsilon/2M \implies |f(x)|\,|g_n - g| < \varepsilon/2.}$

$\leq \mu(\{|f| \geq M\}) + \mu(|g_n - g| \geq \varepsilon/2M)$

$\leq \delta +$; $n \to \infty$ goes to 0.

$\implies \overline{\lim_{n \to \infty}} \mu(|f|\,|g_n - g| \geq \varepsilon/2) \leq \delta$. But true $\forall \delta$.

$\implies \lim_{n \to \infty} \mu(|f|\,|g_n - g| \geq \varepsilon/2) = 0$. $\square$

1st term. Key Lemma is following: Given Key lemma, 1st term is done in the same way

$\forall \delta, \exists M$ s.t. $\mu(|g_n| \geq M) < \delta$ $\forall n$

Inmed. for finitely #.

Choose N_0 s.t. $\mu(|g_n - g| \geq 1) \leq \frac{\delta}{2}$ $n \geq N_0$

~~$\mu(|g_n$~~ Choose M' s.t. $\mu(|g| \geq M') < \frac{\delta}{2}$.

$\mu(|g_n| \geq M' + 1) \leq \mu(\{|g_n - g| \geq 1\} \cup \{|g| \geq M'\}) < \delta$

for $n \geq N_0$ $\square$